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COMPLEMENTED SUBSPACES OF PRODUCTS OF HILBERT SPACES

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ABSTRACT. It is proved that every complemented subspace of an arbitrary topological product of (nonnecessarily separable) Hilbert spaces is isomorphic to a product of Hilbert spaces.

A counterexample is given showing that this result cannot be proved by the same direct method as for countable products.

INTRODUCTION

In the paper [3] the question is asked if every complemented subspace Z of a product of Banach spaces $\prod_{i\in I} X_i$ is necessarily isomorphic to another product of Banach spaces. We solve this question positively in a very specific case when all X_i are Hilbert spaces. In that case Z is a projective limit of a family (Z_j) of Hilbert spaces where linking maps are projections. If the family (Z_j) is countable (i.e. if we consider the countable products of Hilbert spaces) Z must be isomorphic to the product of Hilbert spaces because every projective limit of a sequence of Banach spaces and projections onto is isomorphic to the product of Banach spaces. At the end of the paper we will present a counterexample which shows that the last claim is false for uncountable families of Banach (Hilbert) spaces.

This means that the direct method of the proof of our result fails for arbitrary (uncountable) products of Hilbert spaces. We are forced to use here the highly complicated method similar in spirit to the approach used in [3] to prove that every complemented subspace of a power of $l_p(\Gamma)$, $1 \leq p \leq \infty$, or c_0 contains a large power of l_p or c_0 (see also [6]). The present paper should be considered as a complement to [3]. We emphasize that the crucial role in the solution is played by the strict description of all closed operator ideals (in the sense of Pietsch) in the "algebra" of all operators between finite products of spaces X_i. Since in our case such finite products are simply Hilbert spaces, the description

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mentioned above is known [7, 5.4] as a by-product of the polar decomposition and the spectral representation theorem.

Now, we need some conventions. By $\mathbf{K}$ we denote the scalar field or the one-dimensional locally convex space (lcs). If $X = \prod_{i\in I} X_i$, we identify $\prod_{i\in J} X_i$, $J \subseteq I$, (or X_i) with a subspace of X consisting of families vanishing outside J (or $\{i\}$). We will assume that a 0-neighborhood means absolutely convex one. By $\ker U$ we denote kernel of U, i.e., $\bigcap_{n\in\mathbf{N}}(1/n)U$. We say that $T: E \to F$, E, F are lcs, is an operator if it is linear and continuous. We say that T *fixes a complemented copy of* (an lcs) Y if there is a subspace $G \simeq Y$ of E such that $T|_G$ is an isomorphism and $T(G)$ is complemented in F. As far as the functional analysis terminology is concerned we follow [5].

By gothic letters we denote cardinal numbers and by $\mathfrak{m}^+$ we denote the successor of $\mathfrak{m}$. We define in a slightly specific manner the cofinality $\mathrm{cf}(\mathfrak{m})$ of the cardinal number $\mathfrak{m}$, namely, this will be the infimum of all cardinal numbers $\mathfrak{n}$ such that there is a family $\mathscr{F}$, $\mathrm{card}\,\mathscr{F} \leq \mathfrak{n}$, of cardinal numbers such that $\sup_{\mathfrak{r}\in\mathscr{F}} \mathfrak{r}^+ = \mathfrak{m}$. In particular, $\mathrm{cf}(\mathfrak{m}^+) = 1$.

From now on we assume that $X = \prod_{i\in I} X_i$, where X_i are Hilbert spaces and P is a projection onto a complemented subspace Z of X.

Before we begin our proof, let us recall the result [3, Lemma 2.1 and 3.1] which is the heart of our method:

Lemma 1. *Let* $(I_k)_{k\in K}$ *be a family of infinite sets of the same cardinality,* $\mathrm{card}\, K \leq \mathrm{card}\, I_k$, *and let*

$$Y = \prod_{k\in K} \prod_{i\in I_k} Y_i .$$

If E *is an arbitrary locally convex space and* $T: Y \to E$ *is such an operator that* $T|_{Y_i}$ *is an isomorphism for every* $i \in I_k$, $k \in K$, *and* $T(Y_i)$ *are complemented in* E, *then* T *fixes a complemented copy of* Y.

We will also use Pełczyński's decomposition method in the following form: if X, Y are lcs such that X contains a complemented copy of Y and vice versa, then X is isomorphic to Y whenever X is isomorphic to its countable power $\prod_{i\in\mathbf{N}} X$.

Let us recall [3] that an lcs E has *the property* (A) if there is a bounded set B in E such that for every 0-neighborhood U in E $\ker U + B$ is a 0-neighborhood in E. It is observed [2, Proposition 2(b)] that for Frèchet spaces E the property (A) means exactly that E is a *quojection*, i.e. the projective limit of a sequence of surjective operators and Banach spaces (see [1, p. 590]).

Lemma 2 [3, Lemma 1.1, 1.2, 1.3, and 1.4]. *If* U *is a* 0*-neighborhood in a complemented subspace in an arbitrary product of Banach spaces, then* $\ker U$ *has the property* (A).

Our proof will be divided into two parts. In §1 we define two "cardinal functions" and using them we make the product X containing our complemented subspace Z as "small" as possible. Then, in §2, we prove the theorem by the transfinite induction with respect to those defined cardinal parameters. Section 3 contains the counterexample mentioned above.

1. Auxiliary cardinal functions

Let $T: E \to F$ be an operator between lcs. Then we define two cardinal functions: $w(T)$ is the infimum over all $\mathfrak{m}$ such that there is a family $\mathscr{F}$, $\operatorname{card}\mathscr{F} \leq \mathfrak{m}$, of continuous seminorms on E such that T is still continuous whenever we take on E the topology generated by $\mathscr{F}$ (instead of the original one). Secondly, $sd(T)$ is the infimum over all $\mathfrak{m}$ such that for every bounded set B in E and every 0-neighborhood U in F there is a set A, $\operatorname{card} A < \mathfrak{m}$ (strict inequality!) such that

$$T(A) + U \supseteq T(B).$$

Additionally, we define

$$w(E) := w(id_E), \qquad sd(E) := sd(id_E).$$

It is easily seen that $w(T) < \aleph_0$ iff $w(T) = 1$ iff T sends some 0-neighborhoods into bounded sets (similarly, $sd(T) < \aleph_0$ iff $T = 0$). On the other hand, $w(E) = \aleph_0$ iff E is a nonnormable metrizable lcs and $sd(T) = \aleph_0$ iff T is a nonzero map sending bounded sets into precompact ones. For nontrivial complete lcs E $sd(E) = \aleph_0$ iff E is semi-Montel (see [5, 11.5]).

We will also use the density character $d(E)$ of an lcs E, which is the infimum of all $\mathfrak{m}$ such that there is a dense subset A of E, $\operatorname{card} A \leq \mathfrak{m}$.

The main aim of §1 is the following result:

Theorem 3. *Let $w(Z) = \mathfrak{l}$, $sd(Z) = \mathfrak{m}$. Then for every family of cardinal numbers $(\mathfrak{r}_j)_{j\in J}$, $\mathfrak{r}_j < \mathfrak{m}$, $\operatorname{card} J \leq \mathfrak{l}$, $\sup_{j\in J} \mathfrak{r}_j^+ = \mathfrak{m}$, Z is isomorphic to a complemented subspace of*

$$H_{\mathfrak{l},\mathfrak{m}} := \prod_{j\in J} \left[l_2(\mathfrak{r}_j)\right]^{\mathfrak{l}}.$$

Remark. By Prop. 4(g) below, $\operatorname{cf}(\mathfrak{m}) \leq \mathfrak{l}$, thus the family $(\mathfrak{r}_j)_{j\in J}$ really exists. Moreover, the space $H_{\mathfrak{l},\mathfrak{m}}$ does not depend on the choice of $(\mathfrak{r}_j)$. Indeed, for every choice $sd(H_{\mathfrak{l},\mathfrak{m}}) = \mathfrak{m}$, $w(H_{\mathfrak{l},\mathfrak{m}}) = \mathfrak{l}$ (see Proposition 4(c) and (d) below), thus our theorem implies that they are embedable one into another as complemented subspaces. By Pełczyński's decomposition, they are isomorphic.

In order to prove Theorem 3 we need the following proposition:

Proposition 4. *Let E, F, G, E_i for $i \in I$, be lcs and let $T: E \to F$, $S: F \to G$ be arbitrary operators. Then the following assertions hold:*

(a) $sd(S \circ T) \leq sd(S), sd(T)$; $w(S \circ T) \leq w(S), w(T)$;
(b) *if (T_i) is a net of operators converging uniformly on bounded sets to T and $sd(T_i) \leq \mathfrak{m}$, then $sd(T) \leq \mathfrak{m}$;*
(c) *if $E = \prod_{i\in I} E_i$ and $sd(T|_{E_i}) \leq \mathfrak{m}$ for every $i \in I$, then $sd(T) \leq \mathfrak{m}$;*

(d) *if* $F = \prod_{i\in I} F_i$, $P_i: F \to F_i$ *are the natural projections and* $sd(P_i \circ T) \leq \mathfrak{m}$ *for every* $i \in I$, *then* $sd(T) \leq \mathfrak{m}$;
(e) *if* S *is a topological embedding, then* $sd(S \circ T) = sd(T)$;
(f) *if* T *is a quotient map, then* $w(S \circ T) = w(S)$;
(g) *if* E *has the property* (A), *then* $\mathrm{cf}(sd(E)) \leq w(E)$.

If we assume additionally that E, F, G, E_i *for* $i \in I$, *are normed spaces, then*:

(h) *if* T *is a quotient map, then* $sd(S \circ T) = sd(S)$;
(i) *if* E *is infinite dimensional, then* $d(E)^+ = sd(E)$;
(j) *if the range of* T *is dense in* F, *then* $sd(F) \leq sd(E)$;
(k) *if* $Y = \prod_{i\in I} E_i$, W *is a complemented subspace of* Y *and* $W_1 = W \cap \prod_{i\in J} E_i$, $J \subseteq I$, *then* $w(W/W_1) \leq \mathrm{card}\, I\backslash J$.

Proof. The results (a), (b), (e), (f), and (h) are obvious, (k) is a particular case of [3, 2.9] and (j) is implied by (i). The parts (c) and (d) are simple for a finite set I. To prove them in the whole generality it is enough to recall how 0-neighborhoods and bounded sets in topological products look alike.

(i): The inequality $sd(E) \leq d(E)^+$ holds obviously for every lcs E. Let B be the unit ball of E, there is a subset A of E, $\mathrm{card}\, A < sd(E)$, satisfying

$$B \subseteq \left(\tfrac{1}{2}\right) B + A.$$

The following set

$$A_0 = \left\{ \sum_{i=k}^{l} 2^i x_i : x_i \in A,\ k, l \text{ are integers} \right\}$$

is dense in E. Since $\dim E = \infty$, then $sd(E) > \aleph_0$ and

$$d(E) \leq \mathrm{card}\, A_0 = \aleph_0 \,\mathrm{card}\, A < sd(E).$$

(g): If $w(E) = 1$, then E is a normed space and, by (i),

$$\mathrm{cf}\, sd(E) = \mathrm{cf}(d(E)^+) = 1.$$

Let us assume that $w(E) \geq \aleph_0$, then there is a 0-neighborhood base $(U_i)_{i\in I}$ of E consisting of open sets such that $\mathrm{card}\, I = w(E)$. Let us choose a bounded set B according to the definition of the property (A). Then, for $i \in I$, there are subsets $A_i \subseteq E$ satisfying

$$\mathrm{card}\, A_i < sd(E) \quad \text{and} \quad B \subseteq U_i + A_i.$$

If C is an arbitrary bounded set in E, then

$$C \subseteq nB + \ker U_i \text{ for a suitable chosen } n \in \mathbf{N}.$$

Moreover, if $nU_j \subseteq U_i$, then

$$C \subseteq nB + \ker U_i \subseteq nU_j + nA_j + \ker U_i \subseteq nA_j + U_i,$$

because $U_i + \ker U_i \subseteq U_i$ (U_i is an absolutely convex open set!). Obviously, $\mathrm{card}(nA_i) = \mathrm{card}\, A_i$, hence

$$\sup_{i\in I}(\mathrm{card}\, A_i)^+ = sd(E) \quad \text{and} \quad \mathrm{cf}\, sd(E) \le \mathrm{card}\, I = w(E).$$

Theorem 3 is an immediate consequence of the following strengthened form of [3, Proposition 1.8].

Proposition 5. *Let Y be a complemented subspace of a product $E = \prod_{i\in I} E_i$ of Banach spaces. Then there is a subset $J \subseteq I$, $\mathrm{card}\, J < \aleph_0$ or $\mathrm{card}\, J \le w(Y)$, and there is a family of subspaces $F_i \subseteq E_i$ satisfying*

$$sd(F_i) \le sd(Y) \qquad \text{for } i \in J,$$

such that Y is isomorphic to a complemented subspace of

$$F = \prod_{i\in J} F_i .$$

Proof. According to [3, Prop. 1.8], there is a subset $J \subseteq I$, finite or $\mathrm{card}\, J \le w(Y)$, such that Y is isomorphic to a complemented subspace Y_0 of $E_0 = \prod_{i\in J} E_i$. Let $Q_i\colon E_0 \to E_i$ be the natural projection and let

$$Y_1 = Y_0 \cap \prod_{j\ne i} E_j .$$

Then $Q_i|_{Y_0}$ factorizes through the natural quotient map $q\colon Y_0 \to Y_0/Y_1$, i.e.

$$Q_i = s \circ q , \qquad \text{where } s\colon Y_0/Y_1 \to E_i .$$

By Lemma 2, Y_0 has the property (A). Let B be an absolutely convex bounded subset of Y_0 from the definition of that property. We denote by Y_B the linear span of B in Y_0 equipped with the norm being the gauge functional of B. If $j\colon Y_B \to Y_0$ is the embedding map, then $q \circ j$ is the quotient map and, by Prop. 4(a) and (h),

$$sd(Y_0/Y_1) = sd(q\circ j) \le sd(id_{Y_0}) = sd(Y).$$

Moreover, Prop. 4(j) implies

$$sd\left(\overline{Q_i(Y_0)}\right) = sd\left(\overline{s(Y_0/Y_1)}\right) \le sd(Y_0/Y_1),$$

because $Q_i(Y_0) = s(Y_0/Y_1)$. Now, it is enough to choose

$$F_i = \overline{Q_i(Y_0)} \qquad \text{for } i \in J .$$

2. The inductive proof

We begin with some auxiliary results.

Lemma 6. *Let* $J \subseteq I$, $\operatorname{card} I\backslash J < \operatorname{card} I$. *Then there exists* $J_0 \subseteq J$, $\operatorname{card} I\backslash J = \operatorname{card} I\backslash J_0$, *such that*

$$Z_1 := Z \cap \prod_{i\in J_0} X_i$$

is a complemented subspace of Z.

Proof. First, let us observe that the kernel of any absolutely convex 0-neighborhood U in X is complemented! Indeed, let $\ker U \supseteq \prod_{i\in J} X_i = X_J$, $J \subseteq I$, $\operatorname{card} I\backslash J < \infty$ and let $Q: X/X_J \to X/\ker U$, $P: X \to X/X_J$, $q: X \to X/\ker U$ be the natural quotient maps. Since X/X_J is isomorphic to a Hilbert space and X_J is a complemented subspace of X, there are right inverses S and R for Q and P, respectively. Thus $T: X \to X$

$$T := id - R \circ S \circ q$$

is a projection onto $\ker U$.

Now, if V and W are absolutely convex 0-neighborhoods in Z and in a complement Y of Z, respectively, then for $U = V \oplus W$, $\ker U = \ker V \oplus \ker W$. Since $\ker U$ is complemented in X, $\ker V$ is complemented in Z.

This completes the proof if $\operatorname{card} I\backslash J < \aleph_0$, because then we can choose V in such a way that $\ker V = Z \cap \prod_{i\in J} X_i$.

If $\operatorname{card} I\backslash J \geq \aleph_0$, then our lemma is true even for X_i being a non-Hilbertian Banach space. According to a simple remark after [3, Lemma 1.7] there is a subset $J_1 \subseteq J$ such that

$$\operatorname{card} I\backslash J_1 = \operatorname{card} I\backslash J$$

and

$$P\left(\prod_{i\in J_1} X_i\right) \subseteq \prod_{i\in J} X_i.$$

Thus, we can define inductively a sequence of sets (J_n) satisfying:

$$J_{n+1} \subseteq J_n, \quad \operatorname{card} I\backslash J_n = \operatorname{card} I\backslash J,$$

$$P\left(\prod_{i\in J_{n+1}} X_i\right) \subseteq \prod_{i\in J_n} X_i.$$

If we define

$$J_0 = \bigcap_{n\in\mathbf{N}} J_n,$$

then $\operatorname{card} I\backslash J_0 = \operatorname{card} I\backslash J$ and

$$P\left(\prod_{i\in J_0} X_i\right) \subseteq \prod_{i\in J_0} X_i, \qquad P(Z_1) = Z_1.$$

The previous lemma allows us to make an inductive reduction as follows.

Lemma 7. *If* P *satisfies the following condition: there is a subset* $J \subseteq I$, $\operatorname{card} I\backslash J < \operatorname{card} I = \mathfrak{l}$, *with* $sd(P|_{X_i}) \leq \mathfrak{m}$ *for every* $i \in J$. *Then* $Z \simeq Z_1 \oplus Z_2$, *where*

$$w(Z_2) < \mathfrak{l} \quad \text{and} \quad sd(Z_1) \leq \mathfrak{m}.$$

Proof. According to Lemma 6 there is a set $J_0 \subseteq J$, $\operatorname{card} I\backslash J = \operatorname{card} I\backslash J_0$, such that $Z = Z_1 \oplus Z_2$ (topologically), where

$$Z_1 := Z \cap \prod_{i \in J_0} X_i .$$

By Prop. 4(a), (c), (e), and (k), we get:

$$sd(Z_1) = sd(P|_{Z_1}) \leq \mathfrak{m},$$
$$w(Z_2) = w(Z/Z_1) < \mathfrak{l}.$$

The next lemma is implied by known properties of Hilbert spaces.

Lemma 8. *Every Banach subspace* E *of* X *is complemented and it is a Hilbert space.*

The next point in which we essentially apply properties of Hilbert spaces is the following simple consequence of the description of closed operator ideals on Hilbert spaces [7, 5.4]:

Lemma 9. *Let* Y *be a Hilbert space and let* $T: Y \to X$ *be an arbitrary operator. For every* $\mathfrak{r} < sd(T)$, T *fixes a complemented copy of* $l_2(\mathfrak{r})$.

Proof. First, we prove the particular case when X is a Hilbert space.

Let us prove our result for an operator $S: L_2(\Omega, \mu) \to L_2(\Omega, \mu)$ of the form:

$$S(f) = sf,$$

where $s \in L_\infty(\Omega, \mu)$ and μ is a probability measure. We define:

$$\Omega_n := \{\omega \in \Omega: |s(w)| \geq 1/n\}, \ \mu_n := \mu|_{\Omega_n}.$$

Let $J_n: L_2(\Omega_n, \mu_n) \to L_2(\Omega, \mu)$, $Q_n: L_2(\Omega, \mu) \to L_2(\Omega_n, \mu_n)$ be the standard embedding and projection, respectively. If the Hilbertian dimension of $L_2(\Omega_n, \mu_n)$, i.e. card Γ, where $l_2(\Gamma) \simeq L_2(\Omega_n, \mu_n)$, is not less than $\mathfrak{r}$, then we are done because $S|_{L_2(\Omega_n, \mu_n)}$ is an isomorphism. Let us assume that $\dim(L_2(\Omega_n, \mu_n)) < \mathfrak{r}$ for every $n \in \mathbf{N}$, then

$$\|S - S \circ J_n \circ Q_n\| < 1/n$$

and

$$sd(S \circ J_n \circ Q_n) \leq \mathfrak{r}.$$

Applying Proposition 4(b) we get

$$sd(S) \leq \mathfrak{r}.$$

Now, a simple combination of the polar decomposition and the spectral representation theorem for positive operators [4], [7.D.3] gives us the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & X \\ {\scriptstyle U}\uparrow\downarrow{\scriptstyle U^*} & & {\scriptstyle V}\uparrow\downarrow{\scriptstyle V^*} \\ L_2(\Omega, \mu) & \xrightarrow[S]{} & L_2(\Omega, \mu) \end{array}$$

for some probability space (Ω, μ) and S of the form as above. Moreover, the diagram commutes, i.e.

$$V^* \circ T \circ U = S, \qquad T = V \circ S \circ U^*.$$

This means that either for some $n \in \mathbf{N}$ the Hilbertian dimension of $U(L_2(\Omega_n, \mu_n))$ is not less than $\mathfrak{t}$ and $T_{|U(L_2(\Omega_n, \mu_n))}$ is an isomorphism or $sd(T) \leq sd(S) \leq \mathfrak{t}$ (see Proposition 4(a)). The latter possibility contradicts our assumption.

This completes the proof for X isomorphic to a Hilbert space. In the general case, let $P_i \colon X \to X_i$ be the natural projection. Proposition 4(d) implies that there exists $i \in I$ such that

$$sd(P_i \circ T) > \mathfrak{t}.$$

It is enough to apply the preceding particular case of our theorem to $P_i \circ T$ and Lemma 8.

The last lemma allows us to construct inside a complemented subspace Z of a product of Hilbert spaces a copy of "large" $H_{\mathfrak{l},\mathfrak{m}}$.

Lemma 10. *Let $T \colon X \to X$ be an arbitrary operator and let $\mathfrak{m}$, $\mathfrak{l}$ be cardinal numbers satisfying $\operatorname{cf}\mathfrak{m} \leq \mathfrak{l}$. If for every $\mathfrak{t} \leq \mathfrak{m}$ the inequality $\operatorname{card} J_{\mathfrak{t}} \geq \mathfrak{l}$ holds, where*

$$J_{\mathfrak{t}} := \left\{ i \in I \colon sd\left(T|_{X_i}\right) > \mathfrak{t} \right\},$$

then T fixes a complemented copy of $H_{\mathfrak{l},\mathfrak{m}}$.

Proof. If $\mathfrak{m} = \mathfrak{n}^+$, then $\operatorname{card} J_{\mathfrak{n}} \geq \mathfrak{l}$ and for $i \in J_{\mathfrak{n}}$, $sd(T|_{X_i}) \geq \mathfrak{m}$. According to Lemma 9, for $i \in J_{\mathfrak{n}}$ the map $T|_{X_i}$ fixes a complemented copy of $l_2(\mathfrak{n})$. By Lemma 1, this completes the proof in the case $\mathfrak{m} = \mathfrak{n}^+$.

If $\mathfrak{m}$ is a limit cardinal, then there is a family $(J_k)_{k \in K}$ of subsets of I, where $\operatorname{card} K = \operatorname{cf}\mathfrak{m} \leq \mathfrak{l}$, and a family of cardinal numbers $(\mathfrak{t}_k)_{k \in K}$, $\sup_{k \in K} \mathfrak{t}_k^+ = \mathfrak{m}$, such that

$$sd\left(T|_{X_i}\right) > \mathfrak{t}_k \quad \text{for } i \in J_k \quad \text{and} \quad \operatorname{card} J_k = \mathfrak{l}.$$

Without loss of generality we may assume that J_k are pairwisely disjoint. Thus Lemma 9 implies that for $i \in J_k$ the operator $T|_{X_i}$ fixes a complemented copy of $l_2(\mathfrak{t}_k)$. Finally, Lemma 1 completes the proof.

Theorem 11. *Every complemented subspace of a product* $X = \prod_{i\in I} X_i$ *of Hilbert spaces is isomorphic to a product of Hilbert spaces.*

Proof. We go through the transfinite induction. First, with respect to $sd(Z)$. If $sd(Z) = \aleph_0$, then Proposition 5 implies that $Z \simeq \mathbf{K}^{w(Z)}$ (see [3, Theorem 1.6]).

Let us assume that everything is proved for $sd(Z) < \mathfrak{m}$ and assume that $sd(Z) = \mathfrak{m}$. The inductive step will be proved by the transfinite induction over $w(Z)$. For $w(Z) = 1$ the theorem is trivial. Now, assume our theorem for $w(Z) < \mathfrak{l}$ and let us prove it for $w(Z) = \mathfrak{l}$.

We can assume, according to Theorem 3 and the remarks after it, that Z is contained as a complemented subspace in $X = H_{\mathfrak{l},\mathfrak{m}}$. Let $P\colon X \to Z$ be a projection onto. Then Lemma 10 applied to X and the operator P implies that either Z contains a complemented copy of $H_{\mathfrak{l},\mathfrak{m}}$ or there is $\mathfrak{r} < \mathfrak{m}$ such that

$$\operatorname{card}\left\{i \in I : sd\left(P|_{X_i}\right) > \mathfrak{r}\right\} < \mathfrak{l}.$$

In the first case Pełczyński's decomposition method implies $Z \simeq H_{\mathfrak{l},\mathfrak{m}}$. In the second case we can apply Lemma 7, thus $Z \simeq Z_1 \oplus Z_2$, where

$$sd(Z_1) \leq \mathfrak{r} \quad \text{and} \quad w(Z_2) < \mathfrak{l}.$$

Since the inductive hypothesis is applicable both for Z_1 and Z_2, the proof is complete.

3. The counterexample

We present an example of a projective limit of a family of Hilbert spaces which is not isomorphic to a product of Banach spaces in spite of the fact that linking maps are projections onto!

Let us take $X := l_2(I)$, I uncountable, and let us equip X with the topology generated by the family of seminorms $(p_A)_{A\in\mathscr{C}}$, $\mathscr{C} := \{A : A \subseteq I, \operatorname{card} A \leq \aleph_0\}$:

$$p_A\left((x_i)_{i\in I}\right) := \left(\sum_{i\in A} |x_i|^2\right)^{1/2}.$$

As easily seen, X is the projective limit of the family of Hilbert spaces $(X_A)_{A\in\mathscr{C}}$, $X_A := \{x = (x_i)_{i\in A} : p_A(x) < \infty\}$, where linking maps $T_{AB}\colon X_A \to X_B$, $B \subseteq A$, are projections defined as follows:

$$T_{AB}\left((x_i)_{i\in A}\right) := (x_i)_{i\in B}.$$

On the other hand, X fulfills the following condition $(*)$: for every sequence of continuous seminorms (p_n) on X there is a continuous seminorm p on X such that

$$\bigcap_{n\in\mathbf{N}} \ker p_n \supseteq \ker p.$$

Obviously, no infinite product Y of Banach spaces satisfies $(*)$ and therefore Y cannot be isomorphic to X.

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