PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 110, Number 1, September 1990

# AN INTERPOLATION THEOREM IN SYMMETRIC FUNCTION $F$-SPACES

H. HUDZIK AND L. MALIGRANDA

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. It is well known that every separable or perfect symmetric Banach function space $X$ is an interpolation space between $L^1$ and $L^\infty$ (see [1] and [4]). In this paper we prove that every symmetric function $F$-space is interpolation between $L^0$ and $L^\infty$, where $L^0$ is the space of all measurable functions whose support has finite measure. Moreover, for any function $f \in L^0 + L^\infty$ the norm $\|f\|_{L^0+L^\infty}$ is computed in the terms of the nonincreasing rearrangement function $f^*$ of $f$ as well as in terms of its distribution function $d_f$.

## 1. INTRODUCTION

In the sequel $(I, m)$ will denote the Lebesgue measure space on $I = [0, 1]$ or on $I = [0, \infty)$. By $S = S(I, m)$ we mean the space of all equivalence classes of measurable real-valued functions defined on $I$.

Assume that $E = E(m)$ is a subgroup of $S(I, m)$ endowed with a functional $\| \, \|_E \colon E \to \mathbb{R}_+$ that is even vanishing only at zero and satisfying the triangle inequality. Then $E$ is said to be a *function $F^*$-group.* A function $F^*$-group $(E, \| \, \|_E)$ such that $E$ is a linear space and the operation of multiplication by scalars is continuous is called a *function $F^*$-space.* Any function $F^*$-group ($F^*$-space) that is a complete metric space with respect to the metric $d(f, g) = \|f - g\|_E$ is called a *function $F$-group* ($F$-*space*). For details concerning $F$-spaces see [7].

By $L^0 = L^0(m)$ we shall denote the space of all measurable functions defined on $I$ whose support has finite measure. This space is endowed with the group-norm

$$\|f\|_0 = m(\text{suppf}), \text{ where } \text{suppf} = \{t \in I \colon f(t) \neq 0\}.$$

By $L^\infty = L^\infty(m)$ we shall mean the Banach space of all $m$-essentially bounded functions endowed with the norm $\|f\|_\infty = \text{ess}_{t\in I} \sup |f(t)|$.

Received by the editors March 20, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46E30, 46M35.
*Key words and phrases.* Symmetric function $F$-spaces, interpolation.

The intersection and the algebraic sum of the spaces $L^0$ and $L^\infty$ shall be written as $L^0 \cap L^\infty$ and $L^0 + L^\infty$, respectively. The norms in these spaces will be defined as follows

$$\|f\|_{L^0 \cap L^\infty} = \max\{\|f\|_0, \|f\|_\infty\},$$
$$\|f\|_{L^0 + L^\infty} = \inf\{\|g\|_0 + \|h\|_\infty : f = g + h, \ g \in L^0, \ h \in L^\infty\}.$$

The spaces $L^0$ and $L^0 \cap L^\infty$ are function $F$-groups, and $L^0 + L^\infty$ is a function $F$-space (all are symmetric, see below).

In $f \in S(I, m)$, then the distribution function $d_f$ is defined by

$$d_f(s) = m(\{t \in I : |f(t)| > s\}), \qquad s \geq 0.$$

The nonincreasing *rearrangement* of $f$ is given by

$$f^*(t) = \inf\{s > 0 : d_f(s) \leq t\}, \qquad \inf \phi = \infty.$$

Clearly, $d_f$ and $f^*$ are nonincreasing right-continuous functions on $[0, \infty)$. Notice that if $mI = \infty$, then these functions may also take value $\infty$.

A function $F$-group ($F$-space) $(E, \| \|_E)$ is called *symmetric* (or *rearrangement invariant*) if:

1. $g \in S$, $f \in E$, and $|g| \leq |f|$, a.e., imply $g \in E$ and $\|g\|_E \leq \|f\|_E$,
2. For any $g \in S$ equimeasurable with $f \in E$, i.e., if $d_g = d_f$, then one has $g \in E$ and $\|g\|_E = \|f\|_E$.

Conditions 1 and 2 are equivalent to the following:

1: If $g \in S$, $f \in E$, $g^*(t) \leq f^*(t)$, and all $t \in I$, then $g \in E$ and $\|g\|_E \leq \|f\|_E$. Some properties of symmetric function $F$-spaces have been studied in [3].

## 2. Imbedding theorem and examples

Our first result is related to the problem given at the end of [2].

**Theorem 1.** (a) *If $E$ is a nontrivial symmetric function $F$-group then*

$$L^0 \cap L^\infty \subset E. \tag{1}$$

(b) *If $E$ also satisfies one of the following conditions:*

(i) $\|1_A\|_E < \infty$ *implies* $mA < \infty$, *or*

(ii) $E$ *is a symmetric function $F$-space,*

*then*

$$E \subset L^0 + L^\infty. \tag{2}$$

*Proof.* (a) Let $0 \neq f \in E$. Define $A_n = \{t \in I : |f(t)| \geq \frac{1}{n}\}$, $n = 1, 2, \ldots$. We have $A_n \uparrow$ and $\cup_{n=1}^\infty A_n = \text{supp} f$. Thus, $mA_k > 0$ for some $k \in \mathbb{N}$. Since $k^{-1} 1_{A_k} \leq |f|$, we have $k^{-1} 1_{A_k} \in E$. By the assumption on $E$ we have $1_{A_k} \in E$. Now, let $g \in L^0 \cap L^\infty$. There exist pairwise disjoint sets $B_1, B_2, \ldots, B_\ell$ such that $m(B_i) = m(A_k)$ for $i = 1, 2, \ldots, \ell$ and $\text{supp} g \subset \bigcup_{i=1}^\ell B_i$. All functions

$1_{B_i}(i = 1, 2, \dots, \ell)$ and $1_{A_k}$ are pairwise equimeasurable. Since, in view of our assumption, $E$ is symmetric, we have

$$\|1_{B_1}\|_E = \|1_{B_2}\|_E = \cdots \|1_{B_\ell}\|_E = \|1_{A_k}\|_E.$$

Hence

$$\|1\,\mathrm{suppg}\|_E \le \left\|\sum_{i=1}^{\ell} 1_{B_i}\right\|_E \le \sum_{i=1}^{\ell} \|1_{B_i}\|_E = \ell\|1_{A_k}\|_E < \infty$$

and so $\|g\|_E \le \left\|\|g\|_\infty 1_{\mathrm{suppg}}\right\|_E \le ([\|g\|_\infty] + 1)\|1_{\mathrm{suppg}}\|_E < \infty$, where $[\|g\|_\infty]$ denotes the integer part of the number $\|g\|_\infty$. Thus, $g \in E$ and the inclusion (1) is proved.

(b) Assume first that $E$ satisfies condition (i). Let $f \in E$ and $A = \{t \in I : |f(t)| \le 1\}$. Then $f1_A \in L^\infty$ and

$$\|1_{I\setminus A}\|_E \le \|f1_{I\setminus A}\|_E \le \|f\|_E < \infty.$$

Thus, in view of (i), $m(I\setminus A) < \infty$, i.e., $f1_{I\setminus A} \in L^0$.

Now assume that $E$ satisfies condition (ii) and $f \in E$. Define $A_n = \{t \in I : |f(t)| \ge n\}$, $n = 1, 2, \dots$. We shall prove that $mA_k < \infty$ for some $k \in \mathbb{N}$. In fact, if not, then we have $mA_n = \infty$ for any $n \in \mathbb{N}$. Choose $B_n \subset A_n$ with $mB_n = 1$ for any natural $n$. All functions $1_{B_n}$ are equimeasurable, and so

$$0 \le \|1_{B_1}\|_E = \|1_{B_n}\|_E \le \left\|\frac{1}{n}f1_{B_n}\right\|_E \le \left\|\frac{1}{n}f\right\|_E \to 0 \quad \text{as} n \to \infty.$$

Thus, $\|1_{B_1}\|_E = 0$, a contradiction. Since $f1_{A_k} \in L^0$ and $f1_{I\setminus A_k} \in L^\infty$, the proof is finished.

*Remarks.* (1). O'Neil proved in [5] that the symmetric Orlicz group $L^\psi$ generated by generalized Young function $\psi$ satisfies $L^0 \cap L^\infty \subset L^\psi \subset L^0 + L^\infty$. Moreover, $L^\psi$ is an $F$-space, i.e., the scalar multiplication is continuous in $L^\psi$ if and only if $\lim_{u\to 0} \psi(u) = 0$.

(2). Let us note that in (b) it can be obtained that if $f \in E$ then for every $\varepsilon > 0$ there are functions $f_1 \in L^\infty$ and $f_2 \in L^0$, with $f_1 + f_2 = f$ such that $\|f_1\|_\infty \le \varepsilon$ in the first case and $\|f_2\|_0 \le \varepsilon$ in the second one.

**Example 1.** Let $E = S$ on $I = [0, \infty)$ with the $F$-group norm

$$\|f\|_E = \frac{m(\mathrm{suppf})}{1 + m(\mathrm{suppf})} \qquad \left(\text{taking by definition } \frac{\infty}{1+\infty} = 1\right).$$

Then $E$ does not satisfy any condition (i) and (ii) from Theorem 1, and $E \not\subset L^0 + L^\infty$.

Thus, it is interesting to pose the following problem:

Is the alternative of conditions (i) and (ii) necessary for (2)? Now, we give examples of symmetric function $F$-spaces and $F$-groups connected with Orlicz and Marcinkiewicz spaces.

**Example 2.** Let $\psi$ be an Orlicz function, i.e., a continuous increasing function on $[0, \infty)$ such that $\psi(0) = 0$. The *Orlicz space* $L^\psi$ is the space of all $f \in S$ such that $I_\psi(\lambda f) = \int_I \psi(\lambda|f(t)|)\,\mathrm{dm} < \infty$ for some $\lambda > 0$ depending on $f$. The functional $\|\cdot\|_\psi$ defined on $L^\psi$ by $\|f\|_\psi = \inf\{\lambda > 0 \colon I_\psi(f/\lambda) \le \lambda\}$ is an $F$-norm. Define a new Orlicz function $\psi$ by

$$\psi(u) = \frac{\varphi(u)}{1+\varphi(u)}.$$

Then $L^\psi = L^0 + L^\varphi$ and $\|f\|_\psi \le \|f\|_+ \le 4\|f\|_\psi$, where

$$\|f\|_+ = \inf\{\|f_0\|_0 + \|f_1\|_\varphi \colon f = f_0 + f_1,\ f_0 \in L^0,\ f_1 \in L^\varphi\}.$$

In fact, from the inequality $\psi(u) \le \min(1, \varphi(u))$ we have $L^0 \subset L^\psi$ and $L^\varphi \subset L^\psi$, which imply $L^0 + L^\varphi \subset L^\psi$ and the first inequality on the $F$-norms. On the other hand, if $f \in L^\psi$ and $\|f\|_\psi < \lambda$ then $I_\psi(f/\lambda) \le \lambda$, and if $A = \{t \in I \colon |f(t)| > \lambda\varphi^{-1}(1)\}$, then

$$\frac{mA}{2} \le I_\psi(f1_A/\lambda) \le I_\psi(f/\lambda) \le \lambda.$$

Hence $f1_A \in L^0$ and

$$I_\varphi(f1_{I\setminus A}/\lambda) \le 2I_\psi(f1_{I\setminus A}/\lambda) \le 2I_\psi(f/\lambda) \le 2\lambda,$$

i.e., $f1_{I\setminus A} \in L^\varphi$. Thus $f \in L^0 + L^\varphi$ and $\|f\|_+ \le \|f1_A\|_0 + \|f1_{I\setminus A}\|_\varphi \le mA + 2\lambda \le 4\lambda$.

**Example 3.** Let $\psi$ be an Orlicz function. Define the *Marcinkiewicz space* $M^\psi$ generated by functional

$$\|f\|_{M^\psi} = \inf\{\lambda > 0 \colon \sup_{t>0} \frac{f^*(t)}{\psi^{-1}(\lambda/t)} \le \lambda\}.$$

Then $M^\psi$ is the symmetric function $F$-group. It is sufficient to prove the triangle inequality for $\|\cdot\|_{M^\psi}$. If $\|f\|_{M^\psi} < \lambda_1$ and $\|g\|_{M^\psi} < \lambda_2$, then $f^*(t) \le \lambda_1\psi^{-1}(\lambda_1/t)$ and $g^*(t) \le \lambda_2\psi^{-1}(\lambda_2/t)$ for any $t > 0$. Hence, by the property of rearrangement and the above,

$$\begin{aligned}(f+g)^*(t) &\le f^*\left(\frac{\lambda_1}{\lambda_1+\lambda_2}t\right) + g^*\left(\frac{\lambda_2}{\lambda_1+\lambda_2}t\right)\\ &\le \lambda_1\psi^{-1}\left(\frac{\lambda_1+\lambda_2}{t}\right) + \lambda_2\psi^{-1}\left(\frac{\lambda_1+\lambda_2}{t}\right)\end{aligned}$$

and so $\|f+g\|_{M^\psi} \le \lambda_1 + \lambda_2$, which gives the triangle inequality. Let us note that

$$\|f\|_{M^\psi} = \inf\left\{\lambda > 0 \colon \sup_{s>0} \psi(s/\lambda)d_f(s) \le \lambda\right\}$$

and

$$\|f\|_{M^\psi} \le \|f\|_\psi \quad \text{for } f \in L^\psi.$$

$M^{\psi}$ is a symmetric function $F$-space, i.e., the scalar multiplication is continuous in $M^{\psi}$ if and only if $\psi^{-1}$ satisfies the $\Delta_2$-condition (see [5, Theorem 9.9]). Moreover, $M^{\psi}(0, 1)$ is separable if and only if $\lim_{u\to\infty} \psi(u) = c < \infty$ and $M^{\psi}(0, \infty)$ is not separable (see [5, Theorems 9.14 and 9.15]).

## 3. An interpolation theorem

The first result is closely connected with the approximation spaces introduced by Peetre and Sparr in [6].

**Proposition 1.** *Let $f \in S$ and $0 \le s$, $t < mI$. Then*

$$f^*(t) = \inf\{\|f - g\|_\infty : \|g\|_0 \le t\} = \inf\{\|f - f1_A\|_\infty : mA \le t\} \tag{3}$$

*and*

$$d_f(s) = \inf\{\|f - h\|_0 : \|h\|_\infty \le s\}. \tag{4}$$

*Proof*. Let $f^*(t) < \infty$ and $B = \{x \in I : |f(x)| > f^*(t)\}$. Then $mB = d_f(f^*(t)) \le t$ and

$$\begin{aligned} E(t, f) :&= \inf\{\|f - g\|_\infty : \|g\|_0 \le t\} \\ &\le \inf\{\|f - f1_A\|_\infty : mA \le t\} \\ &\le \|f - f1_B\|_\infty = \|f1_{I\setminus B}\|_\infty \le f^*(t). \end{aligned}$$

If $f^*(t) = \infty$ then $mI = \infty$ and $f \notin L^\infty$. Hence for any $g \in L^0$ with $\|g\|_0 \le t$, we have $f - g \notin L^\infty$ and $E(t, f) = \infty$. Conversely, if $E(t, f) < \infty$, then for any $\varepsilon > 0$ there exists $g \in L^0$ with $\|g\|_0 \le t$ such that $\|f - g\|_\infty < E(t, f) + \varepsilon$. Assuming $\|f - g\|_0 = u$ we get $|f| - |g| \le |f - g| \le u$, a.e. in $I$, and so

$$\{x \in I : |f(x)| > u\} \subset \{x \in I : |g(x)| > 0\} \cup A,$$

where $A \subset I$ is a set of measure zero. Thus,

$$d_f(u) \le \|g\|_0 \le t \quad \text{and} \quad f^*(t) \le u = \|f - g\|_\infty < E(t, f) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof of (3) is complete. If $h \in L^\infty$ is such that $\|h\|_\infty \le s$, then

$$|f(x) - h(x)| \ge |f(x)| - |h(x)| \ge |f(x)| - s \quad \text{a.e. in } I$$

and so

$$A_s = \{x \in I : |f(x)| > s\} \subset \{x \in I : |f(x) - h(x)| > 0\} \cup A,$$

where $A \subset I$ is a set of measure zero. Thus $d_f(s) = mA_s \le \|f - h\|_0$. Moreover, taking $h_0 = f1_{I\setminus A_s}$ we have $\|h_0\|_\infty \le s$ and $\|f - h_0\|_0 = \|f1_{A_s}\|_0 = d_f(s)$. Thus

$$\inf\{\|f - h\|_0 : \|h\|_\infty \le s\} = d_f(s).$$

**Proposition 2.** *Let $E$ be a symmetric function $F$-space.*

(a) *If $A$ is a subset of $I$ of finite measure and $f, f_n \in E$ are such that $f_n \to f$ uniformly in $A \backslash B$, with $mB = 0$, then $\|(f - f_n)1_A\|_E \to 0$ as $n \to \infty$.*

(b) *The set of countable valued real functions is dense in $E$.*

(c) *An operator $\sigma_a (a > 0)$ defined on $E$ by $\sigma_a f(t) = f(t/a)$ is continuous in $E$ and*

$$\|\sigma_a f\|_E \le K_a \|f\|_E$$

*for any $f \in E$ and for a constant $K_a > 0$ independent of $f$.*

*Proof.* (a) From Theorem 1 and the fact that $mA$ is finite it follows that $1_A \in E$. Now, the continuity of the multiplication operation yields

$$\|(f - f_n)1_A\|_E \le \| \|f - f_n\|_\infty 1_A\|_E \to 0 \quad \text{as } n \to \infty.$$

(b) Let $f \in E$ and let $\{I_n\}_{n=1}^\infty$ be a sequence of pairwise disjoint sets of positive finite measure such that $\bigcup_{n=1}^\infty I_n = I$. For a given $\varepsilon > 0$ and $n \in N$ there is a sequence $\{g_m^n\}_{m=1}^\infty$ of countable valued functions with supports in $I_n$ and such that $g_m^n \to f$ uniformly in $I_n \setminus A_n$ where $mA_n = 0$, as $m \to \infty$. Thus, in view of (a), we have

$$\|f 1_{I_n} - g_{m_n}^n\|_E \le 2^{-n}\varepsilon$$

for an $m_n \in \mathbb{N}$, $n = 1, 2, \ldots$. Denote, for brevity, $g_{m_n}^n$ by $g_n$ and define $g = \sum_{n=1}^\infty g_n$. Then

$$\|f - g\|_E = \left\| \sum_{n=1}^\infty (f 1_{I_n} - g_n) \right\|_E \le \sum_{n=1}^\infty \|f 1_{I_n} - g_n\|_E \le \sum_{n=1}^\infty 2^{-n}\varepsilon = \varepsilon.$$

(c) By virtue of (b), the proof proceeds in the same way as the proof of Theorem 4.4 in [4].

**Theorem 2.** *Let $T\colon L^0 + L^\infty$ be a homomorphism, i.e.,*

$$T(f + g) = Tf + Tg \quad \textit{and} \quad T(-f) = -Tf$$

*for any $f, g \in L^0 + L^\infty$. Assume that $T$ maps $L^0$ into $L^0$ and $L^\infty$ into $L^\infty$, and*

$$m(\text{suppTf}) \le M_0 m(\text{suppf}) \qquad \forall f \in L^0, \tag{5}$$

$$\|Tf\|_\infty \le M_1 \|f\|_\infty \qquad \forall f \in L^\infty. \tag{6}$$

*Then $T$ maps any symmetric function $F$-space $E$ into itself and there exists a constant $M > 0$ such that*

$$\|Tf\|_E \le M \|f\|_E \tag{7}$$

*for any $f \in E$.*

*Proof.* First we shall prove that

$$(Tf)^*(M_0 t) \le M_1 f^*(t) \tag{8}$$

for any $f \in L^0 + L^\infty$ and $t > 0$.

For this purpose note that for any $g \in L^0$ with $\|g\|_0 \leq t$ we have $\|Tg\|_0 \leq M_0\|g\|_0 \leq M_0 t$ and, in view of (3)

$$(Tf)^*(M_0 t) \leq \|Tf - Tg\|_\infty = \|T(f-g)\|_\infty \leq M_1\|f-g\|_\infty,$$

i.e.,

$$(Tf)^*(M_0 t) \leq M_1 E(t, f) = M_1 f^*(t).$$

Let $K > 0$ be a constant such that $\|\sigma_{M_0} f\|_E \leq K\|f\|_E$ for any $f \in E$ (see Proposition 2(c)). Since each $f \in E$ belongs to $L^0 + L^\infty$ (see Theorem 1) we can apply inequality (8) and we get $\|Tf\|_E = \|(Tf)^*\|_E \leq K\|\sigma_{1/M_0}(Tf)^*\|_E \leq K\|M_1 f^*\|_E \leq K([M_1]+1)\|f^*\|_E = K([M_1]+1)\|f\|_E$, where $[M_1]$ denotes the integer part of the number $M_1$. The proof is finished.

It is interesting, and useful as well, to describe the space $L^0 + L^\infty$ and the $F$-norms $K(u, f) = \inf\{\|g\|_0 + u\|h\|_\infty : f = g+h,\ g \in L^0,\ h \in L^\infty\}$, $(u > 0)$ of any function $f \in L^0 + L^\infty$ in terms of its distribution function $d_f$ as well as in terms of its nonincreasing rearrangement $f^*$.

**Proposition 3.** *The space $L^0 + L^\infty$ consists of all functions $f$ in $S$ such that $d_f(s) < \infty$ for some $s > 0$. Moreover, we have*

$$K(u, f) = \inf_{s>0}[su + d_f(s)] = \inf_{t>0}[t + uf^*(t)]. \tag{9}$$

*Proof*. Assume that $d_f(s) < \infty$ for some $s > 0$, i.e., $mA_s < \infty$ where $A_s = \{x \in I : |f(x)| > s\}$. We have $f1_{A_s} \in L^0$, $f1_{I\setminus A_s} \in L^\infty$, $f = f1_{A_s} + f1_{I\setminus A_s}$ and so $f \in L^0 + L^\infty$. Now, assume that $f \in L^0 + L^\infty$, i.e., $f = g + h$ with $g \in L^0$ and $h \in L^\infty$. Let $s > 2\|h\|_\infty$. Then

$$\begin{aligned}\{x \in I : |f(x)| > s\} &\subset \{x \in I : |g(x)| + h(x)| > s\} \\ &\subset \{x \in I : |g(s)| > s/2\} \cup \{x \in I : |h(s)| > s/2\} \\ &= \{x \in I : |g(x)| > s/2\} \cup A.\end{aligned}$$

where $mA = 0$. Hence, $d_f(s) \leq d_g(s/2) \leq m(\text{suppg}) < \infty$. If $f \in L^0 + L^\infty$ then for any $\varepsilon > 0$ there exists a decomposition $f = g + h$ such that $\|g\|_0 + u\|h\|_\infty < K(u, f) + \varepsilon$. Let $\|g\|_0 = a$ and $\|h\|_\infty = b$. Then by Proposition 1,

$$\begin{aligned}\inf_{t>0}[t + uf^*(t)] &\leq a + uf^*(a) \\ &= a + uE(a, f) \leq a + u\widetilde{E}(a, f) \leq a + u\|h\|_\infty \\ &= \|g\|_0 + u\|h\|_\infty < K(u, f) + \varepsilon,\end{aligned}$$

where $\widetilde{E}(a, f) = \inf\{\|f-g\|_\infty : \|g\|_0 = a\}$, and

$$\inf_{s>0}[su + d_f(s)] \leq bu + d_f(b) \leq bu + \|g\|_0 = \|g\|_0 + u\|h\|_\infty < K(u, f) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have that $\inf_{t>0}[t + uf^*(t)]$ and $\inf_{s>0}[su + d_f(s)]$ do not exceed $K(u, f)$. On the other hand, if $d_f(s_0) < \infty$, then for $s > s_0$ the measure of the sets $A_s = \{x \in I : |f(x)| > s\}$ is finite and their intersection has measure zero. Therefore $\lim_{s\to\infty} d_f(s) = 0$ and so $f^*(t) < \infty$ for any $t > 0$. Setting $A = \{x \in I : |f(x)| > f^*(t)\}$ we have $mA = d_f(f^*(t)) \leq t$ and

$$K(u, f) \leq \|f1_A\|_0 + u\|f1_{I\setminus A}\|_\infty \leq mA + uf^*(t) \leq t + uf^*(t)$$

for any $t > 0$. Thus

$$K(u, f) \leq \inf_{t>0}[t + uf^*(t)].$$

For any $\varepsilon > 0$ there exists an $s_0 > 0$ such that $s_0 u + d_f(s_0) < \inf_{s>0}[su + d_f(s)] + \varepsilon$. Then

$$K(u, f) \leq \|f1_{A_{s_0}}\|_0 + u\|f1_{I\setminus A_{s_0}}\|_\infty \leq d_f(s_0) + us_0 < \inf_{s>0}[su + d_f(s)] + \varepsilon$$

and (9) is proved.

*Remarks.* (3) Putting together the results in [6, Proposition 4.4] and Proposition 1, we also get the proof of (9).

(4) All our results are also true for arbitrary measure space $(\Omega, \Sigma, \mu)$.

(5) Inequality $E(t, f) \leq \widetilde{E}(t, f)$ always holds and if the measure is non-atomic, then it is possible to prove even the equality $E(t, f) = \widetilde{E}(t, f)$.

## References


1. J. Bergh and J. Löfström, *Interpolation spaces, an introduction*, Springer-Verlag, Berlin, Heidelberg, and New York, 1976.
2. H. Hudzik, *On the smallest and largest Orlicz spaces*, Math. Nachr. **141** (1989), 109–115.
3. B. Kotkowski, *Symmetric spaces* I, II, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **16** (1968), 871–875, 951–954.
4. S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of linear operators*, Nauka, Moscow, 1978; English transl., Amer. Math. Soc., Providence, RI, 1982.
5. R. O'Neil, *Integral transforms and tensor products on Orlicz spaces and* $L(p, q)$*-spaces*, J. Analyse Math. **21** (1968), 1–276.
6. J. Peetre and G. Sparr, *Interpolation of normed Abelian groups*, Ann. Mat. Pura Appl. **92** (1972), 217–262.
7. S. Rolewicz, *Metric linear spaces*, 2nd ed., PWN, Warsaw, 1984.



(H. Hudzik) Institute of Mathematics, University of A. Mickiewicz, Matejki 48/49, 60–769 Poznán, Poland

(H. Hudzik and L. Maligranda) Institute of Mathematics, Polish Academy of Sciences, Mielżyńskiego 27/29, 61–725 Poznán, Poland

(L. Maligranda) Departamento de Matemáticas, Instituto Venezolano de Investigaciones Científicas, Apartado 21827, Caracas 1020–A, Venezuela

(L. Maligranda) Departamento de Matemáticas, Facultad de Ciencias, Universidad Central de Venezuela, Apartado 20513, Caracas 1040–A, Venezuela