PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 110, Number 1, September 1990

A CHARACTERIZATION OF ELLIPSOIDS AND BALLS IN $\mathbf{C}^N$

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(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. The following fact is proved: If D is a smooth bounded domain in $\mathbf{C}^n$ for which Bergman and Szegö projections are equal on smooth harmonic functions then D is a ball.

1. INTRODUCTION AND THE STATEMENTS OF RESULTS

Let D be C^∞-smooth bounded domain in $\mathbf{C}^n$. We shall denote by B the orthogonal projection of $L^2(D)$ onto the space $L^2\,\mathrm{Hol}(D)$ of square-integrable holomorphic functions on D. This projection is called the Bergman projection. Let us now consider the orthogonal projection S of $L^2(\partial D)$ onto the Hardy space $H^2(\partial D)$ of holomorphic functions on D which have square integrable traces on ∂D. This projection is the Szegö projection. Consider now the space of functions harmonic on D and C^∞-smooth on $\overline{D}$. For each such function u we have the well-defined operator $Ru = S(u|_{\partial D}) - Bu$. What can we say about the operator R? If D is strictly pseudoconvex, then R is a smoothing operator defined on harmonic functions (see [3]). If D is a ball in $\mathbf{C}^n$ then $R = 0$. The last fact could be checked via using the explicit formulae for Bergman and Szegö projections (see for example [5]).

The very natural question was asked by J. Burbea: Are the balls the only domains in $\mathbf{C}^n$ for which $R = 0$? The answer is "yes". If $R = 0$ for a domain D then D is a ball.

The aim of the present note is to prove this fact. We shall deal at first with a more general problem: Let φ be a smooth function on ∂D, $\varphi > 0$ on ∂D. Denote by S_φ the weighted Szegö projection, t.m. the projection of $L^2(D)$ onto $H^2(\partial D)$ orthogonal with respect to the scalar product $\langle f, g\rangle\varphi = \int_{\partial D} f\overline{g}\varphi\, d\tau$. For which domains D there exists a weight function φ on ∂D such that $R_\varphi u = S_\varphi(u/_{\partial D}) - Bu = 0$ for each harmonic function u, smooth on $\overline{D}$.

Received by the editors March 30, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32F15, 32H10.

The answer is the following:

Theorem 1. *Let D be a smooth bounded domain in $\mathbf{C}^n$, and let $\varphi > 0$ be a real smooth function on ∂D such that $R_\varphi = 0$.*

Then there exists a positive self-adjoint matrix $A = [a_{ij}]$ with positive trace, a point $a \in \mathbf{C}^n$ and a constant $d > 0$ such that

$$D = \left\{ z \in \mathbf{C}^n : \sum_{\substack{i=1 \\ j=1}}^{n} a_{ij} (z_i = a_i) \left(\bar{z}_j - \bar{a}_j\right) < d \right\},$$

$$\varphi = 2 \left[\sum_{j=1}^{n} \left| \sum_{i=1}^{n} a_{ij} (z_i - a_i)^2 \right| \right]^{1/2} \qquad \text{on } \partial D.$$

It means in particular that D is an ellipsoid. Note that if D and φ are such as in thesis of Theorem 1 then

$$R_\varphi = 0.$$

As a consequence of Theorem 1 we shall obtain the following:

Theorem 2. *Let D be a smooth bounded domain in $\mathbf{C}^n$. If $Ru = S(u/_{\partial D}) - Bu = 0$ for each function u smooth on $\overline{D}$ and harmonic on D then D is a ball.*

2. Proofs

Proof of Theorem 1. Let ρ be the solution of biharmonic Dirichlet problem: $\Delta^2 \rho = 0$, $\rho = 0$ on ∂D, $\partial \rho / \partial n = \varphi$ on ∂D. ($\Delta^2 \varphi = \Delta(\Delta \varphi)$.) Since φ is real, ρ is also a real function. Let h be a harmonic and smooth on $\overline{D}$ function, which is orthogonal to $L^2 \operatorname{Hol}(D)$. This means, that $Bh = 0$ and $S_\varphi h = R_\varphi h + Bh = 0$. We have

$$\int_{\partial D} \bar{h} \varphi \, d\sigma = \int_D \Delta \rho \bar{h} \, dv \qquad \text{by Green's formula}.$$

The integral $\int_{\partial D} \bar{h} \varphi \, d\sigma = \int_{\partial D} 1 \cdot S_{\overline{\varphi(h)}} \varphi \, d\sigma = 0$.

Hence $\int_D \Delta \rho \bar{h} = 0$ for each function h harmonic, smooth on $\overline{D}$ and orthogonal to holomorphic functions. Let P denote the orthogonal projection from $L^2(D)$ onto the space $L^2 \operatorname{Harm}(D)$—the space of square integrable functions, harmonic on D. Let $\theta \in C_0^\infty(D)$. For each i the function $h_i = P[\partial \theta / \partial z_i]$ is harmonic, smooth up to the boundary (see [2]) and orthogonal to $L^2 \operatorname{Hol}(D)$. Since $\Delta \rho$ is harmonic we have

$$\int_D \Delta \rho \cdot \frac{\overline{\partial \theta}}{\partial z_i} dv = \int_D \Delta \rho \overline{h_i} = 0.$$

It implies that $\Delta \rho$ is holomorphic on D. Since $\Delta \rho$ is a real function $\Delta \rho$ must be constant, $\Delta \rho = a > 0$.

$$a \cdot \operatorname{vol}(D) = \int_D \Delta \rho \, dv = \int_{\partial D} \varphi \, d\sigma > 0.$$

It means in particular that ρ is strictly subharmonic. Thus $\rho < 0$ on D. Since D is bounded there exists $a \in D$ such that $\rho(a) = \min_{z \in D}(\rho(z)) = -d$, $d > 0$. By shifting the domain D we can assume that $a = 0$. Let now h be as before a smooth on $\overline{D}$ harmonic function orthogonal to $L^2\,\mathrm{Hol}(D)$. For each $i = 1, \ldots, n$ we have

$$\int_{\partial D} z_i \overline{h} \varphi \, d\sigma = \int_{\partial D} \overline{h} \frac{\partial}{\partial n} \rho z_i = \int_D \Delta(\rho z_i) \cdot \overline{h} = 0.$$

We shall prove that $\Delta(\rho z_i)$ is holomorphic on D. Since

$$\Delta\left(\rho z_i\right) = (\Delta\rho)\, z_i + \varphi \frac{\partial \rho}{\partial \overline{z}_i} = a z_i + \varphi \frac{\partial \rho}{\partial \overline{z}_i} \quad \text{and} \quad \Delta \frac{\partial \rho}{\partial \overline{z}_i} = \frac{\partial}{\partial \overline{z}_i} \Delta \rho = 0$$

the function $\Delta(z\rho_i)$ is harmonic. Thus for $\theta \in C_0^\infty(D)$ and $j = 1, \ldots, n$

$$\int_D \Delta\left(\rho z_i\right) \frac{\overline{\partial \theta}}{\partial z_j} = \int \Delta\left(\rho z_i\right) \overline{h}_j = 0$$

and $\Delta(\rho z_i)$ is holomorphic. Hence for each i the function

$$\frac{\partial \rho}{\partial \overline{z}_i} = \Delta\left(\rho z_i\right) - 4 a z_i$$

is holomorphic too. It implies that $\partial \rho / \partial \overline{z}_i \partial z_j$ is holomorphic for each i, j and $\overline{\partial \rho / \partial \overline{z}_i \partial z_j} = \partial \rho / \partial \overline{z}_j \partial z_i$ is also holomorphic. Hence $\partial \rho / \partial z_i \theta \overline{z}_j = a_{ij}$ is constant on D. Thus ρ is a polynomial of second degree. Since

$$\frac{\partial \rho}{\partial \overline{z}_i}(0) = 0, \quad \rho(0) = -d, \quad \rho = \sum_{ij} a_{ij} z_i \overline{z}_j - d.$$

The fact, that D is bounded implies that the quadratic form $\sum a_{ij} z_i \overline{z}_j$ is strictly positive and the matrix A is positive. $\operatorname{Tr} A = \sum_i a_{ij} = \frac{1}{4} \Delta \rho = a/4 > 0$. Our theorem is proved.

Proof of Theorem 2. We can assume again that $a = 0$. It follows from Theorem 1 that

$$\rho = \sum a_{ij} z_i \overline{z}_j - d \quad \text{and } n$$

$$2 \left[\sum_{j=1}^{n} \left| \sum_{i=1}^{n} a_{ij} z_i \right|^2 \right]^{1/2} = \varphi = 1 \qquad \text{on } \partial D.$$

Then

$$\rho_1(z) = 2 \left[\sum_{j=1}^{n} \left| \sum_{i=1}^{n} a_{ij} z_i \right|^2 \right] - 1$$

is also a defining function for D. Note that $\Delta \rho_1 = r = \text{const}$ on $\mathbf{C}^n$. Then $\rho_0 = c\rho_1 = \rho$ since $\rho - c\rho_1$ is harmonic and vanishes on ∂D.

It means that for every j and i

$$c\sum_{k=1}^{n} a_{ik}\overline{a}_{jk} = a_{ij}.$$

We got a matrix equation $cAA^* = A$. Since A is self-adjoint and invertible, we infer that

$$cA = I.$$

It implies that D is a ball.

Remark 1. Theorem 2 could be also obtained directly from the following result: If D is a smooth bounded domain in $\mathbf{R}^n$ such that there exists a function ρ defined on D for which $\rho = 0$, $\partial\rho/\partial n = 1$ on ∂D, $\Delta\rho = \text{const}$ on D then D is a ball. This result was proved by J. Serrin [6]. For further extensions of it see Weinberger [7] and Molzon [4].

Remark 2. Note that during the proof of Theorem 1 we proved in fact the following statement: Let D be a smooth bounded domain in $\mathbf{C}^n$ and let $(L^2\,\mathrm{Harm}(D))^\perp$ be the space of square integrable harmonic functions on D which are orthogonal to holomorphic functions. Then the space $(L^2\,\mathrm{Harm}(D))^\perp \cap C^\infty(D)$ is dense in $(L^2\,\mathrm{Harm}(D))^\perp$.

The above observation is interesting in view of Barret's example [1]. D. Barret constructed a smooth bounded (non-pseudoconvex) domain for which $L^2\,\mathrm{Hol}(D) \cap C^\infty(\overline{D})$ is not dense in $L^2\,\mathrm{Hol}(D)$. (In fact even $L^p\,\mathrm{Hol}(D)$, $p > 2$ is not dense in $L^2\,\mathrm{Hol}(D)$.) It implies in particular that for such a domain the orthogonal projection from $L^2\,\mathrm{Harm}(D)$ onto $(L^2\,\mathrm{Harm})^\perp$ cannot map $L^2\,\mathrm{Harm}(D) \cap C^\infty(\overline{D})$ into itself. However, the functions smooth up to the boundary are dense in $(L^2\,\mathrm{Harm}(D))^\perp$.

Remark 3. Theorems 1 and 2 remain valid if D is C^α-smooth domain, $\alpha > 4$. It follows from Hölder estimates on biharmonic Dirichlet problem for such domains. In the statements of our theorems we must replace the C^∞-smooth on $\overline{D}$ harmonic functions by functions from $\mathrm{Harm}(D) \cap \Lambda_\beta(\overline{D})\beta = \alpha - 4$.

Problem. Let $\psi > 0$ be a function smooth on D and let B_ψ be a weighted Bergman projection associated with weight ψ.

For which domains there exists functions $\varphi > 0$, $\psi > 0$ smooth on ∂D and D, respectively, such that

$$S_\varphi u = B_\psi u \quad \text{for each } u \in C^\infty(\overline{D}) \cap \mathrm{Harm}(D)?$$

References

1. D. Barret, *Irregularity of the Bergman projection on a smooth bounded domain in* C^2, Ann. of Math. **119** (1984), 431–436.
2. S. Bell, *A duality theorem for harmonic functions*, Michigan Math. J. **29** (1982), 123–128.
3. E. Ligocka, *The Bergman projection on harmonic functions*, Studia Math. **85** (1987), 229–246.

4. R. Molzon, *Integral identities and symmetry*, preprint.

5. W. Rudin, *Function theory in the unit ball in* $\mathbf{C}^n$, Springer-Verlag, Berlin, 1980.

6. J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43** (1971), 304–318.

7. H. Weinberger, *Remark on the preceding paper of Serrin*, Arch. Rational Mech. Anal. **43** (1971), 319–320.

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