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ON MYCIELSKI IDEALS

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ABSTRACT. We investigate relationships between Mycielski ideals in 2^ω generated by different systems. For a fixed Mycielski ideal $\mathfrak{M}$ we study properties of its compact members. For a perfect Polish space X and certain sets $A \subseteq X \times 2^\omega$, the positions of $\{x \in X : A_X \notin \mathfrak{M}\}$ in the Borel and projective hierarchies are established and other section properties are observed.

0. INTRODUCTION

The notation concerning descriptive set theory will be derived from [Mo] and the set-theoretical notation will be standard. For example, for any finite sequence $s \in 2^{<\omega}$, let $[s]$ denote the clopen basic set in 2^ω generated by s. For $A \subseteq X \times 2^\omega$ and $x \in X$, let $A_x = \{c \in 2^\omega : (x, c) \in A\}$. Similarly A^d for $d \in 2^\omega$. A pointclass $\mathscr{R}$ is called *closed under continuous substitution* if, for any perfect Polish spaces X, Y and a continuous function $f: X \to Y$, we have $f^{-1}[E] \in \mathscr{R}(X)$ for all $E \in \mathscr{R}(Y)$. We shall use the symbols $\exists^Y$, $\forall^Y$, $\wedge^\omega$ defined as in [Mo, pp. 25–27 and 44].

Let $K \subseteq \omega$ be an infinite set for which $\omega \backslash K$ is infinite; such a K will be called normal here. For $A \subseteq 2^\omega$, consider the game $\Gamma(A, K)$ with perfect information between two players, I and II. Player I chooses $c_i \in 2 = \{0, 1\}$ if $i \in \omega \backslash K$ and player II chooses c_i if $i \in K$. Player I wins if $c = (c_0, c_1, \dots) \in A$ and player II wins in the opposite case (cf. [My]). A strategy is a function $\sigma: 2^{<\omega} \to 2$. Let STR denote the set of all strategies. This set with the obvious product topology is homeomorphic to the Cantor space 2^ω. For $\sigma, \tau \in \mathrm{STR}$ and a normal K, we define $\sigma *_K \tau = c \in 2^\omega$ as follows

$$c_n = \begin{cases} \sigma(c|n) & \text{if } n \notin K \\ \tau(c|n) & \text{if } n \in K \quad \text{for } n \in \omega. \end{cases}$$

If we replace $\tau(c|n)$ by $d(n)$ (where $d \in 2^\omega$) in the above formula, then $\sigma *_K d$ will be defined. Similarly for $d *_K \tau$. The functions described above by $*_K$ are continuous.

A strategy σ is called winning for player I if $(\forall \tau \in \mathrm{STR})(\sigma *_K \tau \in A)$. Similarly for player II. If one of the players has a winning strategy, the game

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$\Gamma(A, K)$ is called determined. Denote by $V_{\mathrm{II}}(K)$ the family of all sets $A \subseteq 2^\omega$ for which player II has a winning strategy in $\Gamma(A, K)$.

Assume that we have a fixed system $\{K_s\}_{s\in 2^{<\omega}}$ of normal sets such that $K_{s\frown 0} \cap K_{s\frown 1} = \varnothing$ and $K_{s\frown 0} \cup K_{s\frown 1} \subseteq K_s$ for all $s \in 2^{<\omega}$ (where $s\frown i$ denotes the respective extension of s). Let $\mathfrak{M} = \bigcap(V_{\mathrm{II}}(K_s) \colon s \in 2^{<\omega}\}$. This is a σ-ideal introduced by Mycielski in [My]. It fulfills natural invariance conditions. Similar σ-ideals can be constructed if the players choose elements of an arbitrary set X, not necessarily $\{0, 1\}$ (cf. [R]). Most of our results can be extended to this case.

In the sequel, K will denote a normal set.

0.1. **Lemma.** *There is a continuous function $f\colon 2^\omega \to 2^\omega$ such that $f^{-1}[\{d\}] \notin V_{\mathrm{II}}(K)$ for all $d \in 2^\omega$.*

Proof. Let $\omega \backslash K = \{n_0, n_1, \dots\}$ and put $f(c) = (c_{n_0}, c_{n_1}, \dots)$ for $c = (c_0, c_1, \dots) \in 2^\omega$. In each game $\Gamma(f^{-1}[\{d\}], K)$ player I has a winning strategy, thus we get the assertion. □

0.2. **Lemma** (cf. [My], Theorem 9). *If $A \notin V_{\mathrm{II}}(K)$ and the game $\Gamma(A, K)$ is determined, there exists a closed $F \subseteq A$ such that $F \notin V_{\mathrm{II}}(K)$.* □

1. Dependence on generating systems

It is remarked in [R] that the ideal $\mathfrak{M}$ heavily depends on the system $\{K_s\}$. In this section we continue these considerations.

Let us recall some definitions. The pair $\mathfrak{J}$, $\mathfrak{F}$ of ideals in $\mathscr{P}(2^\omega)$ is called *orthogonal* if there are two sets $A \in \mathfrak{J}$ and $B \in \mathfrak{F}$ such that $A \cup B = 2^\omega$. We say that a σ-ideal $\mathfrak{J}$ is *sharper* than a σ-ideal $\mathfrak{F}$ if there are σ-ideals $\mathfrak{J}^*$, $\mathfrak{F}^*$ such that both pairs $\mathfrak{J}$, $\mathfrak{J}^*$ and $\mathfrak{F}$, $\mathfrak{F}^*$ are orthogonal and $\mathfrak{J} \subseteq \mathfrak{F}^* \cap \mathfrak{F}$ (cf. [Ba_1]).

The following question is formulated in [Ba_1]. Does there exist an infinite sequence $(\mathfrak{J}_n\}$ of translation–invariant symmetric σ-ideals with Borel bases, such that $\mathfrak{J}_{n+1}$ is sharper than $\mathfrak{J}_n$?

We give an example of such a sequence of size ω_1.

1.1. **Lemma.** *For each countable family $\{\mathfrak{M}_n \colon n \in \omega\}$ of Mycielski ideals, there exists a Mycielski ideal $\mathfrak{M}$ which is orthogonal to every $\mathfrak{M}_n$.*

Proof. Let $\{K_s^n\}_{s\in 2^{<\omega}}$ be the system generating $\mathfrak{M}_n$.

Claim. For each $L \subseteq \omega$ such that $L \cap K_s^n \neq \varnothing$ for all s and n, there are disjoint sets L_0, $L_1 \subseteq L$ with the same property.

Observe that the sets $L \cap K_s^n$ are in fact infinite, so the claim is clear.

Claim. There exists a system $\{K_t\}_{t\in 2} < \omega$ such that $K_t \cap K_s^n \neq \varnothing$ for all s, t and n.

This is a consequence of the previous claim.

Let $\mathfrak{M}$ be the ideal generated by the system $\{K_t\}$. Then $\mathfrak{M}$ is orthogonal to all $\mathfrak{M}_n$'s (the set $A = \{c \in 2^\omega : (\forall t \in 2^{<\omega})(\exists i \in K_t)(c(i) = 0)\}$ belongs to $\mathfrak{M}$, while its complement belongs to all ideals $\mathfrak{M}_n$). $\square$

1.2. **Lemma.** *For each countable family $\{\mathfrak{M}_n : n \in \omega\}$ of Mycielski ideals, there exists a Mycielski ideal $\mathfrak{M}$ contained in the intersection of $\mathfrak{M}_n$'s.*

Proof. Let $\{K_s^n\}_{s\in 2^{<\omega}}$ be the system generating $\mathfrak{M}_n$. Observe that there exists a system $\{K_t\}_{t\in 2^{<\omega}}$ such that $(\forall n)(\forall s)(\exists t)(K_t \subseteq K_s^n)$. Let $\mathfrak{M}$ be the ideal generated by the system $\{K_t\}$. Then $\mathfrak{M}$ is included in all $\mathfrak{M}_n$'s. $\square$

1.3. **Theorem.** *There exists a sequence $\{\mathfrak{M}_\alpha : \alpha < \omega_1\}$ of Mycielski ideals such that $\alpha < \beta$ that $\mathfrak{M}_\beta$ is sharper than $\mathfrak{M}_\alpha$.*

Proof. We use induction and the previous lemmas. $\square$

Now, we show another fact illustrating the dependence of Mycielski ideals on generating systems. Mycielski in [My] proved that all ideals $\mathfrak{M}$ are orthogonal to the ideal $\mathbf{K}$ of meager sets in 2^ω. However, we have

1.4. **Theorem.** *For each $A \in \mathbf{K}$, there exists a Mycielski ideal $\mathfrak{M}$ such that $A \in \mathfrak{M}$.*

Proof. Let G_n be dense open subsets of 2^ω such that $G_{n+1} \subseteq G_n$ and

$$\bigcap\{G_n : n \in \omega\} \cap A = \varnothing.$$

Construct an increasing sequence $\{t_n : n \in \omega\} \subseteq \omega$ with the following property

$$(\forall n)(\forall s : t_n \to 2)(\exists s' : t_{n+1}\setminus t_n \to 2)([s^\frown s'] \subseteq G_n).$$

Let $\{K_s\}$ be an arbitrary system. Put $K_s' = \bigcup\{t_{n+1}\setminus t_n : n \in K_s\}$. Then $\{K_s'\}$ generates the respective ideal $\square$

Remarks. (a) The above theorem shows that there is no Borel function $f: 2^\omega \to 2^\omega$ such that $f^{-1}[\{d\}] \notin \mathfrak{M}$ for all Mycielski ideals $\mathfrak{M}$ and for all $d \in 2^\omega$ (cf. Lemma 0.1).

(b) The analog of the theorem is false in the case of the Baire space ω^ω since the intersection of all Mycielski ideals is then orthogonal to $\mathbf{K}(\omega^\omega)$ (cf. [R], 4.3).

1.5. **Problems.** (a) Each Mycielski ideal is orthogonal in fact to the intersection of $\mathbf{K}$ with the ideal $\mathbf{L}$ of Lebesgue negligible sets in 2^ω (cf. [My]). Does, for each $A \in \mathbf{L}$, there exist a Mycielski ideal $\mathfrak{M}$ such that $A \in \mathfrak{M}$?

(b) If we assume CH, any two Mycielski ideals $\mathfrak{M}$, $\mathfrak{M}'$ are isomorphic (i.e. there is a 1–1 function f from 2^ω onto 2^ω such that $A \in \mathfrak{M} \equiv f[A] \in \mathfrak{M}'$); cf. [Ba$_2$]. Is CH essential? Are any two ideals $\mathfrak{M}$ and $\mathfrak{M}'$ Borel isomorphic?

2. Compact sets from the ideal

In this section we shall study Mycielski ideals from the point of view of compact sets. Many results in this direction for various ideals can be found in [KLW].

Let X be a perfect Polish space. Let $\mathscr{K}(X)$ denote the space of all compact subsets of X, equipped with the Vietoris topology generated by the subbase consisting of the sets $U(G) = \{F \in \mathscr{K}(X) \colon F \subseteq G\}$, $V(G) = \{F \in \mathscr{K}(X) \colon F \cap G \neq \emptyset\}$ where G denotes an open subset of X. It is well known that $\mathscr{K}(X)$ can be metrized by the Hausdorff metric and forms a perfect Polish space which is compact if X is compact (cf. [Ku]).

From now on, let K be a fixed normal set and $\mathfrak{M}$ a fixed Mycielski ideal generated by $\{K_s\}$.

2.1. **Proposition.** *$V_{\mathrm{II}}(K) \cap \mathscr{K}(2^\omega)$ is open in $\mathscr{K}(2^\omega)$.*

Proof. Let $F \in V_{\mathrm{II}}(K) \cap \mathscr{K}(2^\omega)$. Thus there is a strategy τ winning for player II in $\Gamma(F, K)$. Let $D = \{d *_K \tau \colon d \in 2^\omega\}$. Then D is closed. Put $G = 2^\omega \setminus D$. Thus G is open and $F \in U(G) \subseteq V_{\mathrm{II}}(K) \cap \mathscr{K}(2^\omega)$. □

Since $\mathfrak{M}$ contains the family of all finite sets (cf. [My]) forming a dense subset of $\mathscr{K}(2^\omega)$ (cf. [Ku]), we have

2.2. **Corollary.** *$\mathfrak{M} \cap \mathscr{K}(2^\omega) \in \prod_2^0(\mathscr{K}(2^\omega))$. Consequently $\mathfrak{M} \cap \mathscr{K}(2^\omega)$ is residual in $\mathscr{K}(2^\omega)$.* □

Note that similar properties for Lebesgue negligible sets and meager sets are mentioned in [KLW]; see also [L] where the σ-porous sets are studied from that standpoint. Following the notation of [KLW], remark that $\mathfrak{M} \cap \mathscr{K}(2^\omega)$ is complete-$\prod_2^0$, calibrated and noncontrolled. Moreover, any Borel set which does not belong to $\mathfrak{M}$ contains continuum disjoint compact sets that do not belong to $\mathfrak{M}$. Finally, note that $\mathfrak{M}$ contains "many" compact sets since every perfect set includes a perfect set from $\mathfrak{M}$ (cf. [Ba$_2$]).

2.3. **Lemma.** *If $A \in \mathscr{K}(2^\omega) \cap V_{\mathrm{II}}(K)$ and τ is a winning strategy for the second player in the game $\Gamma(A, K)$, then there is an integer $N > 0$ such that, for each $c \in 2^\omega$ with $(\forall k < N, k \in K)(c(k) = \tau(c|k))$, we have $c \notin A$.*

Proof. Let A and τ be as in our assumptions. Since A is closed, thus for each $d \in 2^\omega$, there is $N > 0$ such that $[d *_k \tau|N]$ is disjoint from A. Let $N(d)$ be the first such N. Then the function $d \mapsto N(d)$ is continuous. The compactness of 2^ω implies that there is N_0 such that $(\forall d \in 2^\omega)(N(d) < N_0)$. This N_0 is good. □

2.4. **Lemma.** *For each $A \in \prod_2^0(X \times 2^\omega)$, the relation $R(x, F) \equiv F \subseteq A_x$ is a $\prod_2^0$ subset of $X \times \mathscr{K}(2^\omega)$.*

Proof. It is enough to show that, for an open $G \subseteq X \times 2^\omega$, the relation $Q(x, F) \equiv F \subseteq G_x$ is open in $X \times \mathscr{K}(2^\omega)$. Indeed, if $Q(x, F)$, then since F is compact, one can find open sets $G_1 \subseteq X$ and $G_2 \subseteq 2^\omega$ such that $\{x\} \times F \subseteq G_1 \times G_2 \subseteq G$. Then $(x, F) \in G_1 \times U(G_2) \subseteq Q$. □

3. Section properties

For perfect Polish spaces X, Y and a family $\mathfrak{F} \subseteq \mathscr{P}(Y)$, consider a mapping $\Phi_{\mathfrak{F}}: \mathscr{P}(X \times Y) \to \mathscr{P}(X)$ given by the formula $\Phi_{\mathfrak{F}}(A) = \{x \in X: A_x \notin \mathfrak{F}\}$ for $A \in \mathscr{P}(X \times Y)$. The mapping $\Phi_{\mathfrak{F}}$ is investigated in the literature for the special case where $\mathfrak{F}$ is a σ-ideal. Two well-known ideals of meager sets and of Lebesgue negligible sets (e.g. in $Y = [0, 1]$), respectively, are regular in the following sense: if $\mathfrak{J}$ is the σ-ideal considered, then, for $\alpha < \omega_1$, we have $\Phi_{\mathfrak{J}}[\sum_\alpha^0(X \times Y)] = \sum_\alpha^0(X)$ (consequently $\Phi_{\mathfrak{J}}[\mathrm{BOREL}(X \times Y)] = \mathrm{BOREL}(X)$) and $\Phi_{\mathfrak{J}}[\sum_1^1(X \times Y)] = \sum_1^1(X)$. These results can found in [Ke], [V], [Bu]. The equation $\Phi_{\mathfrak{J}}[\mathrm{BOREL}(X \times Y)] = \mathrm{BOREL}(X)$ was applied in [Ma] and [MaSr] to selection problems and in [G] to products of ideals. There are σ-ideals which are not regular; Shortt in [Sh] gave examples of σ-ideals $\mathfrak{J}$ satisfying $\Phi_{\mathfrak{J}}[\mathrm{Borel}(X \times Y)] = \sum_1^1(X)$. Our purpose is to establish images under $\Phi_{\mathfrak{M}}$ of certain Borel and projective pointclasses.

It is easy to prove that, for A, $A_n \subseteq X \times 2^\omega$, we have

3.1. **Lemma.** (a) $\Phi_{\mathfrak{M}}(A) = \bigcup\{\Phi_{V_{\mathrm{II}(K_s)}}(A): s \in 2^{<\omega}\}$,

(b) $\Phi_{\mathfrak{M}}(\bigcup\{A_n: n \in \omega\}) = \bigcup\{\Phi_{\mathfrak{M}}(A_n): n \in \omega\}$. □

3.2. **Proposition.** *Let $\mathscr{R}$ be a pointclass closed under continuous substitution. Then the following inclusion holds*

$$(\exists^{2^\omega}\mathscr{R})(X) \cup (\forall^{2^\omega}\mathscr{R})(X) \subseteq \Phi_{V_{\mathrm{II}}(K)}[\mathscr{R}(X \times 2^\omega)].$$

Proof. Let us consider the function $f: 2^\omega \to 2^\omega$ from Lemma 0.1. Let $g = (id, f): X \times 2^\omega \to X \times 2^\omega$. Clearly, g is continuous. Now, if $A \in \mathscr{R}(X \times 2^\omega)$, the let $A^* = g^{-1}[A]$. Obviously, $A^* \in \mathscr{R}(X \times 2^\omega)$. Note that $\Phi_{V_{\mathrm{II}}(K)}(A^*) = \exists^{2^\omega} A$ and $\Phi_{V_{\mathrm{II}}(\omega\setminus K)}(A^*) = \forall^{2^\omega} A$. Moreover, $\exists^{2^\omega} A^* = \exists^{2^\omega} A$ and $\forall^{2^\omega} A^* = \forall^{2^\omega} A$. □

3.3. **Proposition.** *Let $\mathscr{R}$ be a pointclass closed under continuous substitutions. Then $(\exists^{2^\omega}\mathscr{R})(X) \subseteq \Phi_{\mathfrak{M}}[\mathscr{R}(X \times 2^\omega)]$ and $(\forall^{2^\omega}\mathscr{R})(X) \subseteq \Phi_{\mathfrak{M}}[\wedge^\omega\mathscr{R}(X \times 2^\omega)]$.*

Proof. Let $A \in \mathscr{R}(X \times 2^\omega)$. The set A^* constructed in the previous proof for the sets A and $K_{\langle 0\rangle}$ has the property that $\exists^{2^\omega} A = \Phi_{\mathfrak{M}}(A^*)$. Now, let A_s^* be the set A^* constructed in the previous proof for the sets A and $\omega\setminus K_s$. Put $A^{**} = \bigcap\{A_s^*: s \in 2^{<\omega}\} \in \wedge^\omega\mathscr{R}(X \times 2^\omega)$. Then $x \in \forall^{2^\omega} A$ implies $(A^{**})_x = \bigcap\{(A_s^*)_x: s \in 2^{<\omega}\} = 2^\omega$, and $x \notin \forall^{2^\omega} A$ implies $(A^{**})_x \in V_{\mathrm{II}}(K_s)$ for all s. Hence $\Phi_{\mathfrak{M}}(A^{**}) = \forall^{2^\omega} A$. □

3.4. **Proposition.** $\Phi_{\mathfrak{M}}[\sum_1^0(X \times 2^\omega)] = \sum_1^0(X)$.

Proof. For an open $A \subseteq X \times 2^\omega$, the set $\Phi_{\mathfrak{M}}(A)$ is equal to the projection of A onto X. If $B \subseteq X$ is open, then $B = \Phi_{\mathfrak{M}}(B \times 2^\omega)$. □

3.5. **Proposition.** $\Phi_{V_{\mathrm{II}}(K)}[\prod_1^0(X \times 2^\omega)] \subseteq \prod_1^0(X)$.

Proof. Let $A \subseteq X \times 2^\omega$ be closed and assume that $x_0 \notin \Phi_{V_{\mathrm{II}}(K)}(A)$. Then $A_{x_0} \in V_{\mathrm{II}}(K)$, so let τ be a winning strategy for player II in the game $\Gamma(A, K)$. Let N be chosen for τ by Lemma 2.3. Put $D = \{c \in 2^\omega : (\forall k < N, k \in K)(c(k) = \tau(c|k))\}$. Obviously, D is a clopen subset of 2^ω, disjoint from A_{x_0}. For each $c \in D$, we can find open sets $U_c \subseteq X$ and $V_c \subseteq 2^\omega$ such that $x_0 \in U_c$, $c \in V_c$ and $(U_c \times V_c) \cap A = \emptyset$. The compactness of D implies that there are $c_1, \dots, c_n \in D$ such that $D \subseteq V = V_{c_1} \cup \dots \cup V_{c_n}$. Let $U = U_{c_1} \cap \dots \cap U_{c_n}$. Then $\{x_0\} \times D \subseteq U \times V$ and $(U \times V) \cap A = \emptyset$. Moreover, for each $x \in U$, τ is a winning strategy for player II in the game $\Gamma(A_x, K)$. Hence $U \cap \Phi_{V_{\mathrm{II}}(K)}(A) = \emptyset$. □

From Lemma 3.1(a) we get

3.6. **Corollary.** $\Phi_{\mathfrak{M}}[\prod_1^0(X \times 2^\omega)] \subseteq \sum_2^0(X)$. □

Remark. If X is a σ-compact space, then the corollary can be proved directly in the following way. Let $A \in \prod_1^0(X \times 2^\omega)$. Then

$$x \in \Phi_{\mathfrak{M}}(A) \equiv (\exists K \in \mathscr{K}(2^\omega))(K \subseteq A_x \wedge K \notin \mathfrak{M}).$$

The relation $K \subseteq A_x \wedge K \notin \mathfrak{M}$ is a $\sum_2^0$ subset of $X \times \mathscr{K}(2^\omega)$. Since $X \times \mathscr{K}(2^\omega)$ is σ-compact, the projection of a $\sum_2^0$ set is a $\sum_2^0$ set.

3.7. **Proposition.** $\sum_2^0(X) \subseteq \Phi_{\mathfrak{M}}[\prod_1^0(X \times 2^\omega)]$.

Proof. Let us assume that $A = \bigcup\{A_n : n \in \omega\}$ where A_n's are closed subsets of X. Let $0 \in 2^\omega$ be the zero-sequence and $B = (X \times \{0\}) \cup \bigcup\{A_n \times [0|n] : n \in \omega\}$. Then B is a closed subset of $X \times 2^\omega$, and $\Phi_{\mathfrak{M}}(B) = A$. □

From Lemma 3.1(b) and the above propositions we get

3.8. **Theorem.** $\Phi_{\mathfrak{M}}[\prod_1^0(X \times 2^\omega)] = \Phi_{\mathfrak{M}}[\sum_2^0(X \times 2^\omega)] = \sum_2^0(X)$. □

3.9. **Proposition.** $\Phi_{\mathfrak{M}}[\prod_2^0(X \times 2^\omega)] \subseteq \sum_1^1(X)$.

Proof. Let $A \in \prod_2^0(X \times 2^\omega)$. Note that

$$A_x \notin \mathfrak{M} \equiv (\exists F \in \mathscr{K}(2^\omega))(F \subseteq A_x \wedge F \notin \mathfrak{M}).$$

The relation $F \subseteq A_x \wedge F \notin \mathfrak{M}$ is Δ_3^0 by Lemma 2.4 and Corollary 2.2. Hence the relation $A_x \notin \mathfrak{M}$ is of type $\sum_1^1$. □

3.10. **Theorem.**

(a) $\Phi_{V_{\mathrm{II}}(K)}[\prod_2^0(X \times 2^\omega)] = \sum_1^1(X)$,

(b) $\Phi_{\mathfrak{M}}[\prod_2^0(X \times 2^\omega)] = \Phi_{\mathfrak{M}}[\sum_3^0(X \times 2^\omega)] = \sum_1^1(X)$,

(c) $\prod_1^1(X) \subseteq \Phi_{V_{\mathrm{II}}(K)}[\sum_2^0(X \times 2^\omega)] \subseteq \Delta_2^1(X)$,

(d) $\prod_1^1(X) \cup \sum_1^1(X) \subseteq \Phi_{\mathfrak{M}}[\prod_3^0(X \times 2^\omega)] \subseteq \Phi_{\mathfrak{M}}[\mathrm{BOREL}(X \times 2^\omega)] \subseteq \Delta_2^1(X)$.

Proof. For (a), use Proposition 3.2 and a slight modification of Proposition 3.9. For (b), use Propositions 3.3, 3.9 and Lemma 3.1. The first inclusion of (c) is a consequence of Proposition 3.2. The second inclusion is a consequence of Borel determinacy (cf. [Mar]): for a Borel set $A \subseteq X \times 2^{\omega}$, we have $A_x \notin V_{\mathrm{II}}(K) \equiv (\forall\tau)(\exists\sigma)((x, \sigma *_K \tau) \in A) \equiv (\exists\sigma)(\forall\tau)((x, \sigma *_K \tau) \in A)$. In the first inclusion of (d) use (b) and Proposition 3.3, in the last use the proof of (c). □

3.11. **Theorem.**

(a) $\Phi_{V_{\mathrm{II}}(K)}[\sum_n^1(X \times 2^{\omega})] = \prod_{n+1}^1(X)$,

(b) $\Phi_{\mathfrak{M}}[\sum_n^1(X \times 2^{\omega}] = \prod_{n+1}^1(X)$,

(c) $\sum_{n+1}^1(X) \subseteq \Phi_{V_{\mathrm{II}}(K)} \prod_n^1(X \times 2^{\omega})] \subseteq \prod_{n+2}^1(X)$,

(d) $\sum_{n+1}^1(X) \subseteq \Phi_{\mathfrak{M}}[\prod_n^1(X \times 2^{\omega})] \subseteq \prod_{n+2}^1(X)$.

Under $\prod_n^1$ *determinacy we can get* $\sum_{n+1}^1(X)$ *on the right sides of* (c) *and* (d).

Proof. Propositions 3.2 and 3.3 give that

$$\prod_{n+1}^{1}(X) \subseteq \Phi_{V_{\mathrm{II}}(K)}\left[\sum_{n}^{1}(X \times 2^{\omega})\right] \cap \Phi_{\mathfrak{M}}\left[\sum_{n}^{1}(X \times 2^{\omega})\right] \quad \text{and}$$

$$\sum_{n+1}^{1}(X) \subseteq \Phi_{V_{\mathrm{II}}(K)}\left[\prod_{n}^{1}(X \times 2^{\omega})\right] \cap \Phi_{\mathfrak{M}}\left[\prod_{n}^{1}(X \times 2^{\omega})\right].$$

For the remaining inclusions, note that if $A \in \sum_n^1(X \times 2^{\omega})$, then $A_x \notin V_{\mathrm{II}}(K) \equiv (\forall\tau)(\exists\sigma)((x, \sigma *_K \tau) \in A)$. The right-side formula presents a $\prod_{n+1}^1$ subset of X. Hence $\Phi_{\mathfrak{M}}(A)$ and $\Phi_{V_{\mathrm{II}}(K)}(A)$ are $\prod_{n+1}^1$. Similarly, if $A \in \prod_n^1(X \times 2^{\omega})$ then the relations $A_x \notin \mathfrak{M}$ and $A_x \notin V_{\mathrm{II}}(K)$ are of type $\prod_{n+2}^1$. If we assume $\prod_n^1$ determinacy, these relations become of type $\sum_{n+1}^1$ by changing the places of the quantifiers. □

3.12. **Problem.** Establish $\Phi_{V_{\mathrm{II}}(K)}[\mathrm{BOREL}(X \times 2^{\omega})]$ and $\Phi_{\mathfrak{M}}[\mathrm{BOREL}(X \times 2^{\omega})]$.

Finally, let us observe that $\mathfrak{M}$ has not the Fubini property.

3.13. **Proposition.** *There exists a set* $A \in \prod_2^0(2^{\omega} \times 2^{\omega})$ *such that all vertical sections* A_c *do not belong to* $\mathfrak{M}$, *while all horizontal sections* A^d *constitute a basis of* $\mathfrak{M}$.

Proof. Let $\pi: 2^{<\omega} \to \omega$ be a bijection, and let $\{L_i : i \in \omega\}$ be a family of disjoint infinite subsets of $K_{\langle 0\rangle}$. For $d \in 2^{\omega}$, $i \in \omega$ and $s \in 2^{<\omega}$, put $(d)_i(s) = d(l_{\pi(s)}^i)$ where $L_i = \{l_c^i, l_1^i, \dots\}$. Let us define $(c, d) \in A \equiv (\forall s)(\exists j)(j \in K_s \wedge (d)_{\pi(s)}(c|j) \neq c(j))$. Clearly, $A \in \prod_2^0(2^{\omega} \times 2^{\omega})$ and the horizontal sections A^d form a basis. Moreover, A_c's do not belong to $\mathfrak{M}$ since player II has no winning strategy in $\Gamma(A_c, K_{\langle 1\rangle})$. □

REFERENCES

[Ba_1] M. Balcerzak, *The decomposition property of σ-ideals* II, Rad. Mat. **3** (1987), 261–266.

[Ba_2] ——, *On σ-ideals having perfect members in all perfect sets*, preprint.

[Bu] J. P. Burgess, *Classical hierarchies from a modern standpoint. Part* I. *C-sets*, Fund. Math. **115** (1983), 81–95.

[G] M. Gavalec, *Iterated products of ideals of Borel sets*, Colloq. Math. **50** (1985), 39–52.

[Ke] A. S. Kechris, *Measure and category in effective descriptive set theory*, Ann. Math. Logic **5** (1973), 337–384.

[KLW] A. S. Kechris, A. Louveau and W. H. Woodin, *The structure of σ-ideals of compact sets*, Trans. Amer. Math. Soc. **301** (1987), 263–288.

[Ku] K. Kuratowski, *Topology,* vols. I and II, Academic Press, New York, 1966, 1968.

[L] L. Larson, *Typical compact sets in the Hausdorff metric are porous*, Real Anal. Exchange **13** (1987–88), 116–118.

[Ma] A. Maitra, *Selectors for Borel sets with large sections*, Proc. Amer. Math. Soc. **89** (1983), 705–708.

[MaSr] A. Maitra and V. V. Srivatsa, *Parametrizations of Borel sets with large sections*, Proc. Amer. Math. Soc. **93** (1985), 545–548.

[Mar] D. A. Martin, *Borel determinacy*, Ann. of Math. **102** (1975), 363–371.

[Mo] Y. N. Moschovakis, *Descriptive set theory*, North-Holland, Amsterdam, 1980.

[My] J. Mycielski, *Some new ideals of sets on the real line*, Colloq. Math. **20** (1969), 71–76.

[R] A. Rosłanowski, *On game ideals*, preprint.

[Sh] R. M. Shortt, *Product sigma—ideals*, Topology Appl. **23** (1986), 279–290.

[V] R. L. Vaught, *Invariant sets in topology and logic*, Fund. Math. **82** (1974), 269–294.

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