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ON PAVING SEQUENCES IN C^*-ALGEBRAS

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ABSTRACT. The notion of a paving sequence in a C^*-algebra is introduced and several properties of C^*-algebras which admit such sequences are studied.

1. INTRODUCTION

In this article we examine a property which reflects a well-behaved interplay between projections and isometries in a separable C^*-algebra. To elaborate, we consider the following definitions.

Let $\mathscr{A}$ be a C^*-algebra. We denote by $\mathscr{A}_{sa}^1$ and $\mathscr{A}_+^1$ the self-adjoint and positive parts of the unit ball of $\mathscr{A}$, respectively. A *tower* of $\mathscr{A}$ is a triple $(\varphi, (p_n), \{u_i\})$ consisting of a state φ of $\mathscr{A}$, a sequence (p_n) $(n = 1, \dots)$ of projections in $\mathscr{A}$ satisfying $p_n p_{n+1} = p_{n+1}$ for each n, and a sequence (u_i) $(i = 1, \dots)$ of isometries in $\mathscr{A}$ (or $\mathscr{A}^\sim$, if $\mathscr{A}$ does not have an identity) such that:

(i) $\varphi(p_n) = 1$ for each n; and
(ii) $\|p_n u_j^* u_i p_n\| \underset{n}{\to} 0$ whenever $i \neq j$.

We shall say that a separable C^*-algebra $\mathscr{A}$ has a *simple paving structure* if there is a tower $(\varphi, (p_n), \{u_i\})$ of $\mathscr{A}$, a dense sequence (a_s) $(s = 1, \dots)$ in $\mathscr{A}_{sa}^1$, and a nondecreasing sequence of positive integers (k_n) $(n = 1, \dots)$ with the following properties:

(P1) $\|p_n u_j^* u_i p_n\| < k_n^{-6} \cdot n^{-1}$ for each pair of distinct i and j in $\{1, \dots, k_n\}$;
(P2) $\|p_n u_j^* a_s u_i p_n - \varphi(u_j^* a_s u_i) p_n\| < k_n^{-3} \cdot n^{-1}$ for all $i, j \in \{1, \dots, k_n\}$ and all $s \in \{1, \dots, n\}$;
(P3) the sequence $(\sum_{i=1}^{k_n} u_i p_n u_i^* : n \in \mathbf{N})$ converges to the identity of the von Neumann algebra $\mathscr{A}^{**}$ in the strong-operator topology.

In this case the sequence $(p_n, \{u_i\}_{i=1}^{k_n} : n \in \mathbf{N})$ will be called a *paving sequence* with support φ.

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Simple examples of paving sequences are obtained by considering uniformly hyperfinite (UHF) C^*-algebras [6]. Suppose $\mathscr{A}$ is a UHF C^*-algebra. Regarding $\mathscr{A}$ as an infinite tensor product $\bigotimes_{r\geq 1} M_{m(r)}$ of full matrix algebras $M_{m(r)}$, let $k_n = m(1)\cdots m(n)$ and $M_{k_n} = M_{m(1)} \otimes \cdots \otimes M_{m(n)}$ $(n = 1, \ldots)$. Let $(e_{ij}^{(r)})$ $(i, j = 1, \ldots, m(r))$ denote the set of matrix units in $M_{m(r)}$. Define the sequence of projections (p_n) in $\mathscr{A}$ by $p_n = e_{11}^{(1)} \otimes e_{11}^{(2)} \otimes \cdots \otimes e_{11}^{(n)} \otimes \mathrm{id}_{n+1}$, where id_{n+1} denotes the identity of $\bigotimes_{r\geq n+1} M_{m(r)}$. Let φ_r be the pure state on $M_{m(r)}$ given by $\varphi_r(\cdot) = \mathrm{tr}_r(\cdot\, e_{11}^{(r)})$ (tr_r denotes the tracial state on $M_{m(r)}$), and φ be the pure product state $\bigotimes_{r\geq 1} \varphi_r$ on $\mathscr{A}$. Then $\varphi(p_n) = 1$ for each n and $p_n a p_n = \varphi(a) p_n$ for each $a \in M_{k_n}$. Further, for each r let v_r be a unitary of order $m(r)$ in $M_{m(r)}$ which permutes diagonal projections of $M_{m(r)}$ (so that $\sum_{p=0}^{m(r)-1} v_r^p e_{11}^{(r)} (v_r^p)^* = \mathrm{id}(M_{m(r)})$). The lexicographic enumeration of the set $\{(p_1, \ldots, p_r, \ldots, p_n, 0, 0, \ldots) : n \in \mathbf{N};\ p_r \in \{0, \ldots, m(r)-1\}$ for $1 \leq r \leq n\}$ induces the enumeration of the set

$$\{v_1^{p_1} \otimes \cdots \otimes v_r^{p_r} \otimes \cdots \otimes v_n^{p_n} \otimes \mathrm{id}_{n+1} : n \in \mathbf{N};\ p_r \in \{0, \ldots, m(r)-1\} \text{ for } 1 \leq r \leq n\}$$

into a sequence of unitaries (u_i) such that $u_i \in M_{k_n}$ for $1 \leq i \leq k_n$ and $u_i p_n$ $(1 \leq i \leq k_n)$ are partial isometries in M_{k_n} with pairwise orthogonal range projections summing up to the identity I of $\mathscr{A}$. Hence, $p_n u_j^* u_i p_n = 0$ for each distinct i and j in $\{1, \ldots, k_n\}$ and $\sum_{i=1}^{k_n} u_i p_n u_i^* = I$ for each n. Since the norm-one self-adjoint elements of $\bigcup_{n\geq 1} M_{k_n}$ are dense in $\mathscr{A}_{\mathrm{sa}}^1$, a standard set-theoretic argument shows that there is a dense sequence (a_s) $(s = 1, \ldots)$ in $\mathscr{A}_{\mathrm{sa}}^1$ such that $a_s \in M_{k_n}$ for each $s \in \{1, \ldots, n\}$. This implies that $p_n u_j^* a_s u_i p_n = \varphi(u_j^* a_s u_i) p_n$ for all i and j in $\{1, \ldots, k_n\}$ and all $s \in \{1, \ldots, n\}$. Consequently, $(p_n, \{u_i\}_{i=1}^{k_n} : n \in \mathbf{N})$ is a paving sequence of $\mathscr{A}$. A similar but, in general, more technically involved procedure may be used to construct paving sequences in matroid separable C^*-algebras [2]. Thus, the main features of C^*-algebras with simple paving structure resemble those of matroid separable C^*-algebras, but the property of the existence of a generating nest of matrix algebras is relaxed.

In this article we study several general properties of C^*-algebras with simple paving structure. We show that the identity map on a C^*-algebra $\mathscr{A}$ containing a paving sequence $(p_n, \{u_i\}_{i=1}^{k_n} : n \in \mathbf{N})$ with support φ is the point-strong limit of the sequence of finite rank completely positive maps T_n on $\mathscr{A}$ given by $T_n(a) = \sum_{i,j=1}^{k_n} \varphi(u_j^* a u_i) u_j p_n u_i^*$. As a consequence, $\mathscr{A}$ is simple; and in the unital case it is nuclear. Another consequence is that the sequence $m_n = \sum_{i=1}^{k_n} u_i p_n \otimes p_n u_i^*$ $(n = 1, \ldots)$ in the algebraic tensor product $\mathscr{A} \otimes \mathscr{A}$ has the property analogous to that of a reduced virtual diagonal, in the sense that $V(am_n) - V(m_n a) \underset{n}{\to} 0$ for each V in the Haagerup dual of $\mathscr{A} \otimes \mathscr{A}$ and all $a \in \mathscr{A}$ (see [4, 5]). This property is used to show that under certain conditions a correspondence between a paving sequence of $\mathscr{A}$ and a similar sequence of another C^*-algebra $\mathscr{B}$ induces a * homomorphism from $\mathscr{A}$ into $\mathscr{B}$.

2. Basic properties

Theorem 1. *Let $(p_n, \{u_i\}_{i=1}^{k_n} : n \in \mathbf{N})$ be a paving sequence with support φ of a C^*-algebra $\mathscr{A}$. Consider the sequence of (finite rank completely positive) maps $T_n : \mathscr{A} \to \mathscr{A}$ given by*

$$T_n(a) = \sum_{i,j=1}^{k_n} \varphi(u_j^* a u_i) u_j p_n u_i^* \qquad (a \in \mathscr{A}, n \in \mathbf{N}).$$

Then $T_n(a) \underset{n}{\to} a$ in the strong-operator topology on $\mathscr{A}^{**}$ for each $a \in \mathscr{A}$.

The proof is based on the following two lemmas.

Lemma 2. *The sequence (T_n) is bounded.*

Proof. Let (b_t) be an increasing approximate identity of $\mathscr{A}$ in the positive part of the unit ball of $\mathscr{A}$, which is quasi-central for $\mathscr{A}^\sim$. By [8, 3.1.4],

$$\|T_n\| = \lim_t \|T_n(b_t)\| = \left\| \sum_{i,j=1}^{k_n} \overline{\varphi}(u_j^* u_i) u_j p_n u_i^* \right\| \quad \text{for each } n,$$

where $\overline{\varphi}$ denotes the canonical extension of φ to $\mathscr{A}^\sim$ given by $\overline{\varphi}(c) = \lim_t \varphi(b_t^{1/2} c b_t^{1/2})$. Since $\varphi(p_n) = 1$, $\overline{\varphi}(c) = \varphi(p_n c p_n)$ for all $c \in \mathscr{A}^\sim$ and all n. In particular, $|\overline{\varphi}(u_j^* u_i)| = |\varphi(p_n u_j^* u_i p_n)| = 0$ when $i \neq j$, since the norms $\|p_n u_j^* u_i p_n\|$ become arbitrarily small for all sufficiently large n. Consequently, $\|T_n\| = \|\sum_{i=1}^{k_n} u_i p_n u_i^*\|$. But the sequence $(\sum_{i=1}^{k_n} u_i p_n u_i^* : n \in \mathbf{N})$ is strongly convergent in $\mathscr{A}^{**}$ and hence is bounded, by the Uniform Boundedness principle. Therefore $\sup\{\|T_n\| : n \in \mathbf{N}\} < \infty$.

Lemma 3. *Let (a_s) be a dense sequence in $\mathscr{A}_{\mathrm{sa}}^1$ which satisfies property* (P2) *of the paving sequence $(p_n, \{u_i\}_{i=1}^{k_n} : n \in \mathbf{N})$. Then for any vector ζ in the universal Hilbert space $\mathscr{H}_u$ of $\mathscr{A}$, any positive integer p and any $\varepsilon > 0$ there is a positive integer N such that $\|a_s\zeta - T_n(a_s)\zeta\| < \varepsilon$ $(s = 1, \dots, p)$ for all $n \geq N$.*

Proof. We may assume that $\|\zeta\| = 1$. Since

$$\begin{aligned} \|a_s\zeta - T_n(a_s)\zeta\|^2 &= \langle (a_s - T_n(a_s))^2\zeta, \zeta\rangle \\ &= \langle (a_s^2 - T_n(a_s)a_s)\zeta, \zeta\rangle - \langle (a_s T_n(a_s) - (T_n(a_s))^2)\zeta, \zeta\rangle, \end{aligned} \tag{1}$$

it suffices to show that for each $s \in \{1, \dots, p\}$ the absolute value of each summand of the right side of (1) is less than $\frac{1}{2}\varepsilon^2$ for all sufficiently large n.

Since the sequence $(\sum_{i=1}^{k_n} u_i p_n u_i^* : n \in \mathbf{N})$ strongly converges to the identity of $\mathscr{A}^{**}$, for any $\delta > 0$ there is a positive integer N_1 such that

$$\left\| \sum_{i=1}^{k_n} u_i p_n u_i^* \zeta - \zeta \right\| < \delta \quad \text{for all } n \geq N_1.$$

This and the Cauchy–Schwarz inequality imply that

(2)

$$\left|\langle (a_sT_n(a_s)-(T_n(a_s))^2)\zeta,\zeta\rangle - \left\langle (a_sT_n(a_s)-(T_n(a_s))^2)\sum_{i=1}^{k_n} u_ip_nu_i^*\zeta,\ \sum_{i=1}^{k_n} u_ip_nu_i^*\zeta\right\rangle\right|$$
$$< M\delta^2 + 2M\delta(1+\delta) \qquad \text{for all } s\in\{1,\dots,p\} \text{ and all } n\geq N_1,$$

where $M=\sup\{\|a_sT_n(a_s)-(T_n(a_s))^2\|: s\in\{1,\dots,p\},\ n\in\mathbf{N}\}$ (and $M<\infty$, from Lemma 2).

On the other hand, for each s and n,

(3)

$$\left\langle (a_sT_n(a_s)-(T_n(a_s))^2)\sum_{i=1}^{k_n} u_ip_nu_i^*\zeta,\ \sum_{i=1}^{k_n} u_ip_nu_i^*\zeta\right\rangle$$
$$= \left\langle \left(\sum_{k=1}^{k_n} u_kp_nu_k^*\zeta\right) \times \left(\sum_{i,j=1}^{k_n}\varphi(u_j^*a_su_i)a_su_jp_nu_i^* - \left(\sum_{q,p=1}^{k_n}\varphi(u_p^*a_su_q)u_pp_nu_q^*\right.\right.\right.$$
$$\left.\left.\left.\times\left(\sum_{i,j=1}^{k_n}\varphi(u_j^*a_su_i)u_jp_nu_i^*\right)\right)\times\left(\sum_{m=1}^{k_n}u_mp_nu_m^*\right)\zeta,\ \zeta\right\rangle$$
$$= S_1+S_2-S_3,$$

where:

$$S_1 = \sum_{k,i,j=1}^{k_n}\varphi(u_j^*a_su_i)\langle u_k(p_nu_k^*a_su_jp_n-\varphi(u_k^*a_su_j)p_n)u_i^*\zeta,\ \zeta\rangle;$$

$$S_2 = \sum_{\substack{k,i,j,m=1\\ i\neq m}}^{k_n}\varphi(u_j^*a_su_i)\langle u_kp_nu_k^*a_su_jp_nu_i^*u_mp_nu_m^*\zeta,\ \zeta\rangle;$$

$$S_3 = \sum_{k,p,q,i,j,m=1}^{k_n}\varphi(u_p^*a_su_q)\varphi(u_j^*a_su_i)\langle u_kp_nu_k^*u_pp_nu_q^*u_jp_nu_i^*u_mp_nu_m^*\zeta,\ \zeta\rangle,$$

provided that the sum is taken over all ordered collections (k,p,q,i,j,m) in $\{1,\dots,k_n\}$ which satisfy at least one of the following conditions: $k\neq p$, $q\neq j$, $i\neq m$.

Let N_2 be a positive integer such that $N_2\geq\max\{p,\delta^{-1}\}$. Counting the number of summands in the sums defining S_1, S_2, and S_3 and applying the estimates of (P1) and (P2) it is easy to see that for each $s\in\{1,\dots,p\}$ and all $n\geq N_2$,

$$\begin{aligned}
&|S_1| < k_n^3\cdot(k_n^{-3}\cdot n^{-1}) = n^{-1}\leq\delta;\\
(4)\qquad &|S_2| < k_n^4\cdot(k_n^{-6}\cdot n^{-1}) \leq n^{-1}\leq\delta;\\
&|S_3| < k_n^6\cdot(k_n^{-6}\cdot n^{-1}) = n^{-1}\leq\delta.
\end{aligned}$$

Letting $N_3 = \max\{N_1, N_2\}$ we obtain from (2), (3), and (4) that for each $s \in \{1, \dots, p\}$ and all $n \geq N_3$,

$$|\langle (a_s T_n(a_s) - (T_n(a_s))^2)\zeta, \zeta\rangle| < M\delta^2 + 2M\delta(1+\delta) + 3\delta,$$

which is less than $\frac{1}{2}\varepsilon^2$ if δ is chosen sufficiently small. Similarly one can show that $|\langle (a_s^2 - T_n(a_s)a_s)\zeta, \zeta\rangle| < \frac{1}{2}\varepsilon^2$ for all $s \in \{1, \dots, p\}$ and all sufficiently large n. This completes the proof of the lemma.

Proof of the Theorem 1. It suffices to show that $\|a\zeta - T_n(a)\zeta\| \underset{n}{\to} 0$ for each $a \in \mathscr{A}_{\mathrm{sa}}^1$ and each vector ζ in the universal Hilbert space $\mathscr{H}_u$. Given such a, ζ and $\varepsilon > 0$, choose an element a_t of the dense sequence (a_s) such that $\|a - a_t\| < \varepsilon$. From Lemma 3, $\|(a_t - T_n(a_t))\zeta\| < \varepsilon$ for all sufficiently large n. Hence,

$$\|a\zeta - T_n(a)\zeta\| \leq \|(a-a_t)\zeta\| + \|(a_t - T_n(a_t))\zeta\| + \|(T_n(a-a_t)\zeta\| < (2+L)\cdot\|\zeta\|\cdot\varepsilon$$

for all sufficiently large n, where $L = \sup\{\|T_n\| : n \in \mathbf{N}\}$. This completes the proof of the theorem.

Corollary 4. *If a C^*-algebra $\mathscr{A}$ has a simple paving structure, then $\mathscr{A}$ is simple.*

Proof. Let $(p_n, \{u_i\}_{i=1}^{k_n} : n \in \mathbf{N})$ be a paving sequence of $\mathscr{A}$ with support φ, and suppose $\mathscr{J}$ is a proper closed two-sided ideal of $\mathscr{A}$. If $\mathscr{J} \neq 0$, then $\mathscr{J}_+^1$ contains a nonzero element c_0. From Theorem 1, $T_n(c_0) \underset{n}{\to} c_0$ in the strong-operator topology on $\mathscr{A}^{**}$, and the maps T_n are the compositions of the completely positive maps $S_n : \mathscr{A} \to M_{k_n}$ and $R_n : M_{k_n} \to \mathscr{A}$ given by

$$S_n(a) = [\varphi(u_j^* a u_i)]; \qquad R_n([\lambda_{ij}]) = \sum_{i,j=1}^{k_n} \lambda_{ij} u_j p_n u_i^* \qquad (i, j = 1, \dots, k_n)$$

([1, Lemma 2.1]). This implies, in particular, that $S_n(c_0) > 0$ for all sufficiently large n, so that $\varphi(u_i^* c_0 u_i) > 0$ for some i. Let $\lambda = \varphi(u_i^* c_0 u_i)$ and $b_n = p_n u_i^* c_0 u_i p_n$. Then $b_n \in \mathscr{J}_+^1$ for each n. From the property (P2) of the paving sequence we can conclude that $\|\lambda^{-1} b_n - p_n\| \underset{n}{\to} 0$. In particular, $\|\lambda^{-1} b_k - p_k\| < \frac{1}{2}$ for some k; and the standard perturbation result (see, e.g., [3, A8.1; A8.2]) shows that there is a projection e in the C^*-subalgebra generated by b_k, and a partial isometry v in $\mathscr{A}$ such that $p_k = v^* e v$. Therefore, p_k (and each p_n with $n \geq k$) belongs to $\mathscr{J}$. But then the sequence $(\sum_{i=1}^{k_n} u_i p_n u_i^* : n \geq k)$ is contained in $\mathscr{J}$, which contradicts property (P3) of the paving sequence, since there are states of $\mathscr{A}$ vanishing on $\mathscr{J}$. Consequently $\mathscr{J} = 0$, and the corollary follows.

Corollary 5. *If a unital C^*-algebra $\mathscr{A}$ has a simple paving structure, then $\mathscr{A}$ is nuclear.*

Proof. Theorem 1 and the Hahn–Banach Theorem imply that the identity map on $\mathscr{A}$ belongs to the point-norm closure of the set of finite rank completely positive maps on $\mathscr{A}$. From [7, Lemma 4], the identity map on $\mathscr{A}$ is a point-norm limit of the set of finite rank completely positive contractions on $\mathscr{A}$. The assertion then follows from [1, Theorem 3.1].

3. Some applications

Recall that the *Haagerup norm* on the algebraic tensor product $\mathscr{B} \otimes \mathscr{C}$ of C^*-algebras $\mathscr{B}$ and $\mathscr{C}$ is defined by

$$\|z\|_h = \inf\left\{ \left\| \sum_{i=1}^{k} b_i b_i^* \right\|^{1/2} \cdot \left\| \sum_{i=1}^{k} c_i^* c_i \right\|^{1/2} : z = \sum_{i=1}^{k} b_i \otimes c_i \right\}.$$

This is indeed a norm, as verified in [5]. From [5, Theorem 2.1] the elements V of the normed dual space $(\mathscr{B} \otimes \mathscr{C}, \|\ \|_h)^*$ (abbreviated as $(\mathscr{B} \otimes \mathscr{C})_h^*$) are characterized by the following property:

$$|V(b \otimes c)| \leq K \rho(bb^*)^{1/2} \cdot \omega(c^* c)^{1/2} \qquad (b \in \mathscr{B}, c \in \mathscr{C})$$

for some states $\rho \in \mathscr{B}^*$, $\omega \in \mathscr{C}^*$ and some constant K. From this it is easy to see that $(\mathscr{B} \otimes \mathscr{C})_h^*$ is a $(\mathscr{C}, \mathscr{B})$-bimodule when left and right multiplications are defined by

$$cV(x \otimes y) = V(x \otimes yc) \quad \text{and} \quad Vb(x \otimes y) = V(bx \otimes c) \qquad (x \in \mathscr{B}, y \in \mathscr{C}).$$

Suppose $(p_n, \{u_i\}_{i=1}^{k_n} : n \in \mathbf{N})$ is a paving sequence of a C^*-algebra $\mathscr{A}$ and consider the sequence (m_n) in $\mathscr{A} \otimes \mathscr{A}$ defined by $m_n = \sum_{i=1}^{k_n} u_i p_n \otimes p_n u_i^*$ $(n = 1, \ldots)$. From the uniform boundedness of the norms $\|\sum_{i=1}^{k_n} u_i p_n u_i^*\|$, this sequence is bounded in the Haagerup norm.

Proposition 6. *If $(p_n, \{u_i\}_{i=1}^{k_n} : n \in \mathbf{N})$ is a paving sequence of a C^*-algebra $\mathscr{A}$ and $m_n = \sum_{i=1}^{k_n} u_i p_n \otimes p_n u_i^*$, then $V(am_n) - V(m_n a) \underset{n}{\rightarrow} 0$ for each V in $(\mathscr{A} \otimes \mathscr{A})_h^*$ and all $a \in \mathscr{A}$.*

Proof. Suppose $a \in \mathscr{A}$, $V \in (\mathscr{A} \otimes \mathscr{A})_h^*$, and let (T_n) be the sequence of finite rank completely positive maps on $\mathscr{A}$ defined in Theorem 1. Choosing the states ρ, $\omega \in \mathscr{A}^*$ and $K > 0$ such that $|V(x \otimes y)| \leq K\rho(xx^*)^{1/2} \cdot \omega(y^* y)^{1/2}$

for all $x, y \in \mathscr{A}$, we have

$$
\begin{aligned}
&|V((a-T_n(a))m_n)| \\
&\quad = \left| V\left(\sum_{i=1}^{k_n}(a-T_n(a))u_ip_n \otimes p_nu_i^*\right)\right| \\
&\quad = \left|\sum_{i=1}^{k_n} V((a-T_n(a))u_ip_n \otimes p_nu_i^*)\right| \\
&\quad \le K\sum_{i=1}^{k_n}[\rho((a-T_n(a))u_ip_nu_i^*(a^*-T_n(a^*)))]^{1/2}[\omega(u_ip_nu_i^*)]^{1/2} \\
&\quad \le K\left[\sum_{i=1}^{k_n}\rho((a-T_n(a))u_ip_nu_i^*(a^*-T_n(a^*)))\right]^{1/2}\left[\sum_{i=1}^{k_n}\omega(u_ip_nu_i^*)\right]^{1/2} \\
&\quad = K\left[\rho((a-T_n(a))\sum_{i=1}^{k_n}u_ip_nu_i^*(a^*-T_n(a^*)))\right]^{1/2}\left[\omega\left(\sum_{i=1}^{k_n}u_ip_nu_i^*\right)\right]^{1/2},
\end{aligned}
$$

which is arbitrarily small for all sufficiently large n, since the sequence $((a^*-T_n(a^*)) : n \in \mathbf{N})$ is strongly convergent to zero and the sequences $(\sum_{i=1}^{k_n} u_ip_nu_i^* : n \in \mathbf{N})$, $((a-T_n(a)) : n \in \mathbf{N})$ are bounded. In the same way, $|V(m_n(a-T_n(a)))|$ is arbitrarily small for all sufficiently large n.

On the other hand for each n,

(5)

$$
\begin{aligned}
T_n(a)m_n &= \left(\sum_{i,j=1}^{k_n}\varphi(u_j^*au_i)u_jp_nu_i^*\right)\left(\sum_{k=1}^{k_n}u_kp_n\otimes p_nu_k^*\right) \\
&= \sum_{i,j=1}^{k_n}\varphi(u_j^*au_i)u_jp_n\otimes p_nu_i^* + \sum_{\substack{i,j,k=1\\ i\neq k}}^{k_n}\varphi(u_j^*au_i)u_jp_nu_i^*u_kp_n\otimes p_nu_k^*,
\end{aligned}
$$

and similarly

$$
(6)\quad m_nT_n(a) = \sum_{i,j=1}^{k_n}\varphi(u_j^*au_i)u_jp_n\otimes p_nu_i^* + \sum_{\substack{i,j,k=1\\ j\neq k}}^{k_n}\varphi(u_j^*au_i)u_kp_n\otimes p_nu_k^*u_jp_nu_i^*.
$$

From (5), (6), and the property (P2) of the paving sequence we can conclude that

$$
(7)\qquad \|T_n(a)m_n - m_nT_n(a)\| < 2\|a\|L^{1/2}n^{-1},
$$

where $L = \sup\{\|\sum_{i=1}^{k_n} u_ip_nu_i^*\| : n \in \mathbf{N}\}$. (Note that the estimate (7) is still valid under the weaker assumption $\|p_nu_j^*u_ip_n\| < k_n^{-3/2}\cdot n^{-1}$ $(i, j \in \{1, \ldots, k_n\}$;

$i \neq j$)). Thus, $|V(T_n(a)m_n) - V(m_nT_n(a))|$, $|V(am_n - T_n(a)m_n)|$ and $|V(m_na - T_n(a)m_n)|$ are arbitrarily small for all sufficiently large n, from which the assertion of the proposition follows.

Proposition 7. *Let $\mathscr{A}$, $\mathscr{B}$ be C^*-algebras and $(p_n, \{u_i\}_{i=1}^{k_n} : n \in \mathbf{N})$ be a paving sequence of $\mathscr{A}$ with support φ. Suppose that there is a sequence (q_n) of projections in $\mathscr{B}$ and a sequence (v_i) of isometries in $\mathscr{B}$ (or $\mathscr{B}^{\sim}$, if $\mathscr{B}$ does not have an identity) such that:*

(i) $\|q_nv_j^*v_iq_n\| < k_n^{-3/2} \cdot n^{-1}$ *for each pair of distinct i and j in $\{1, \dots, k_n\}$.*

(ii) *The sequence $(\sum_{i=1}^{k_n} v_iq_nv_i^* : n \in \mathbf{N})$ converges to the identity of the von Neumann algebra $\mathscr{B}^{**}$ in the strong-operator topology.*

Suppose further that for each $a \in \mathscr{A}$ the sequence

$$\Phi_n(a) = \sum_{i,j=1}^{k_n} \varphi(u_j^*au_i)v_jq_nv_i^* \qquad (n = 1, \dots)$$

converges in the strong-operator topology on $\mathscr{B}^{**}$ to the element $\Phi(a)$ of $\mathscr{B}$, and $\Phi(u_ip_n) = v_iq_n$ for each i and n. Then the map $\Phi: \mathscr{A} \to \mathscr{B}$ is a * homomorphism.

Proof. The map Φ is the point-strong limit of the sequence of completely positive maps Φ_n, from which we conclude that Φ is linear and positive. It suffices, therefore, to show that Φ is multiplicative.

Consider the sequence $m_n = \sum_{i=1}^{k_n} u_ip_n \otimes q_nv_i^*$ $(n = 1, \dots)$ in $\mathscr{A} \otimes \mathscr{B}$. We observe first that

(8) $\quad V(am_n) - V(m_n\Phi(a)) \underset{n}{\to} 0 \qquad$ for each $V \in (\mathscr{A} \otimes \mathscr{B})_h^*$ and all $a \in \mathscr{A}$.

This follows with only minor changes from the proof of Proposition 6. Indeed, given $a \in \mathscr{A}$ and $V \in (\mathscr{A} \otimes \mathscr{B})_h^*$, the first paragraph of the proof of Proposition 6 shows that $V((a - T_n(a))m_n) \underset{n}{\to} 0$ and $V(m_n(\Phi_n(a) - \Phi(a))) \underset{n}{\to} 0$, where $T_n(a) = \sum_{i,j=1}^{k_n} \varphi(u_j^*au_i)u_jp_nu_i^*$. Furthermore, following (5), (6), and (7), we have

$$\|T_n(a)m_n - m_n\Phi_n(a)\| < \|a\|(M^{1/2} + L^{1/2}) \cdot n^{-1},$$

where $M = \sup\{\|\sum_{i=1}^{k_n} v_iq_nv_i^*\| : n \in \mathbf{N}\}$ and $L = \sup\{\|\sum_{i=1}^{k_n} u_ip_nu_i^*\| : n \in \mathbf{N}\}$. Therefore $V(T_n(a)m_n - m_n\Phi_n(a)) \underset{n}{\to} 0$, and (8) follows.

Suppose $a, c \in \mathscr{A}$ and $V \in (\mathscr{A} \otimes \mathscr{B})_h^*$. From the bimodule property of $(\mathscr{A} \otimes \mathscr{B})_h^*$ (see the paragraph following the proof of Corollary 5) the functionals Va and $\Phi(c)V$ belong to $(\mathscr{A} \otimes \mathscr{B})_h^*$. Therefore, from (8),

$$V(acm_n) - V(am_n\Phi(c)) = Va(cm_n) - Va(m_n\Phi(c)) \underset{n}{\to} 0$$

and

$$V(am_n\Phi(c)) - V(m_n\Phi(a)\Phi(c)) = \Phi(c)V(am_n) - \Phi(c)V(m_n\Phi(a)) \underset{n}{\to} 0.$$

Hence $V(acm_n) - V(m_n\Phi(a)\Phi(c)) \underset{n}{\to} 0$.

Also $V(acm_n) - V(m_n\Phi(ac)) \underset{n}{\to} 0$, from (8). Consequently,

$$V(m_n(\Phi(ac) - \Phi(a)\Phi(c))) \underset{n}{\to} 0 \qquad \text{for each } V \in (\mathscr{A} \otimes \mathscr{B})_h^*. \tag{9}$$

Let ψ be a state of $\mathscr{B}$. Define the linear functional V_ψ on $\mathscr{A} \otimes \mathscr{B}$ by

$$V_\psi(x \otimes y) = \psi(\Phi(x)y) \qquad (x \in \mathscr{A}, y \in \mathscr{B}).$$

From the complete positivity of the maps Φ_n, $\Phi_n(x^*)\Phi_n(x) \leq \Phi_n(x^*x)$ for each $x \in \mathscr{A}$ and all n [9]. Since Φ is the point-strong limit of the sequence (Φ_n), the same inequalities will hold for Φ. Therefore,

$$\begin{aligned} |V_\psi(x \otimes y)| = |\psi(\Phi(x)y)| &\leq \psi(\Phi(x)\Phi(x)^*)^{1/2}\psi(y^*y)^{1/2} \\ &\leq \psi(\Phi(xx^*)^{1/2}\psi(y^*y)^{1/2}, \end{aligned}$$

which shows that V_ψ belongs to $(\mathscr{A} \otimes \mathscr{B})_h^*$. Hence, from (9) and the assumption $\Phi(u_ip_n) = v_iq_n$, we have

$$\begin{aligned} &\psi\left(\sum_{i=1}^{k_n} v_iq_nv_i^*(\Phi(ac) - \Phi(a)\Phi(c))\right) \\ &= \psi\left(\sum_{i=1}^{k_n} \Phi((u_ip_n)q_nv_i^*(\Phi(ac) - \Phi(a)\Phi(c))\right) \\ &= V_\psi\left(\sum_{i=1}^{k_n} u_ip_n \otimes q_nv_i^*(\Phi(ac) - \Phi(a)\Phi(c))\right) \underset{n}{\to} 0. \end{aligned}$$

Because the sequence $(\sum_{i=1}^{k_n} v_iq_nv_i^* : n \in \mathbf{N})$ strongly converges to the identity in the von Neumann algebra $\mathscr{B}^{**}$, the last implication shows that

$$\psi(\Phi(ac) - \Phi(a)\Phi(c)) = 0.$$

Therefore $\Phi(ac) - \Phi(a)\Phi(c) = 0$, and Φ is a * homomorphism.

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