PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 110, Number 1, September 1990

GENERIC HOMEOMORPHISMS HAVE THE PSEUDO-ORBIT TRACING PROPERTY

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(Communicated by Kenneth R. Meyer)

ABSTRACT. Let M ($\dim M \leq 3$) be a compact manifold. Then a generic $f \in \mathrm{Homeo}(M)$ satisfies the following: f has the pseudo-orbit tracing property; f is C^0 tolerance stable; and f is not topologically stable.

1. INTRODUCTION

Let M be a compact differentiable manifold with the metric d induced by a Riemannian structure. d. We denote by $\mathrm{Homeo}(M)$ the space of all homeomorphisms on M with the C^0 topology, i.e. the topology induced by the following metric.

$$d(f, g) = \max_{x \in M} d(f(x), g(x)).$$

The C^0 topology depends on neither the differentiable structure nor the Riemannian structure.

A subset in a topological space is called residual if it includes a countable intersection of open and dense subsets. A topological space is called a Baire space if every residual set is dense in it. In particular, every complete metric space is a Baire space. For example, $\mathrm{Homeo}(M)$ is a Baire space because the metric $\tilde{d}$ below, which induces the C^0 topology, makes $\mathrm{Homeo}(M)$ a complete metric space.

$$\tilde{d}(f, g) = d(f, g) + d(f^{-1}, g^{-1}).$$

Moreover, the C^0 closure of all diffeomorphisms, denoted $\mathrm{ClDiff}(M)$, is also a Baire space.

Let P be a property for homeomorphisms. We say that generic homeomorphisms in $\mathrm{Homeo}(M)$ (resp. $\mathrm{ClDiff}(M)$) satisfy P if the set of homeomorphisms satisfying P is residual in $\mathrm{Homeo}(M)$ (resp. $\mathrm{ClDiff}(M)$). There are some results about generic homeomorphisms, for example [5, 12], and the following theorem.

Received by the editors May 30, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58F10; Secondary 54H20.
Key words and phrases. Generic homeomorphisms, POTP, tolerance stability, topological stability.

Theorem ([13]). *If $M = S^1$, then a generic $f \in \mathrm{Homeo}(M)$ satisfies the following*:

(1) *f has the pseudo-orbit tracing property*;
(2) *f is C^0 tolerance stable*; *and*
(3) *f is not topologically stable.*

In §3, we extend the above theorem to the case when $\dim M \leq 3$. The main tool is Shub's density theorem [6].

2. Definitions

After this, f and g denote elements of $\mathrm{Homeo}(M)$.

(1) A sequence $\{x_i\}_{i\in\mathbf{Z}}$ is a δ-pseudo-orbit of f if $d(f(x_i), x_{i+1}) \leq \delta$ for every $i \in \mathbf{Z}$.
(2) A sequence $\{x_i\}_{i\in\mathbf{Z}}$ is ϵ-traced by the f-orbit through $x \in M$ if $d(f^i(x), x_i) \leq \epsilon$ for every $i \in \mathbf{Z}$.
(3) A sequence $\{x_i\}_{i\in\mathbf{Z}}$ is ϵ-set-traced by the f-orbit through $x \in M$ if $\overline{d}(\mathrm{Cl}\{f^i(x) \colon i \in \mathbf{Z}\}, \mathrm{Cl}\{x_i \colon i \in \mathbf{Z}\}) \leq \epsilon$.
   Here we denote by $\mathrm{Cl}\{*\}$ the closure and by $\overline{d}$ the Hausdoff metric with respect to d.
(4) f is strongly C^0 tolerance stable if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for every $g \in V_\delta(f)$, every f-orbit is ϵ-traced by some g-orbit and every g-orbit is ϵ-traced by some f-orbit.
   Here we denote by $V_\delta(f)$ the δ-neighborhood of f in $\mathrm{Homeo}(M)$.
(5) f is C^0 tolerance stable if, for every $\epsilon > 0$; there exists $\delta > 0$ such that, for every $g \in V_\delta(f)$, every f-orbit is ϵ-set-traced by some g-orbit and every g-orbit is ϵ-set-traced by some f-orbit.
(6) f has the pseudo-orbit tracing property (abbr. POTP) if, for every $\epsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo-orbit of f is ϵ-traced by some f-orbit.
(7) f is lower semi-conjugate to g under φ if there exists a continuous surjection $\varphi \colon M \to M$ satisfying $f\varphi = \varphi g$.
(8) f is topologically stable if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for every $g \in V_\delta(f)$, f is lower semi-conjugate to g under φ satisfying $d(\varphi, 1_M) \leq \epsilon$.

By the definition, the topological stability implies the strong C^0 tolerance stability.

Proposition 1. *Let f be strongly C^0 tolerance stable. Then*:

(1) *f is C^0 tolerance stable*; *and*
(2) *f has the POTP.*

Proof.

(1) It is immediate by the fact that the ϵ-traceability implies the ϵ-set-traceability.

(2) The m-dimensional cases ($m \geq 2$) are proved by modification of the proof of [10, Theorem 11]. The 1-dimensional case is also valid, see Remark 2 below.

Remark 2. There is the necessary and sufficient condition that $f \in \mathrm{Homeo}(S^1)$ has the POTP, see [13]. It is also equivalent to the strong C^0 tolerance stability, see [4].

3. Main theorem

Theorem A. *A generic* $f \in \mathrm{ClDiff}(M)$ *satisfies the following:*

(0) f *is strongly* C^0 *tolerance stable;*
(1) f *has the pseudo-orbit tracing property;*
(2) f *is* C^0 *tolerance stable; and*
(3) f *is not topologically stable.*

By the above theorem, for every compact differentiable manifold M, we can see the existence of the homeomorphisms which have the POTP but are not topologically stable.

By [2] or [11], if $\dim M \leq 3$, then $\mathrm{ClDiff}(M) = \mathrm{Homeo}(M)$. Therefore we obtain the following theorem as a corollary.

Theorem B. *If* dim $M \leq 3$, *then a generic* $f \in \mathrm{Homeo}(M)$ *satisfies the properties* (0)–(3) *in Theorem* A.

4. Proof of Theorem A

The combination of Shub's density theorem [6] (or [7]) and Nitecki's topological stability theorem [3] implies the following lemma.

Lemma 3. *The set of all topologically stable homeomorphisms is dense in* $\mathrm{ClDiff}(M)$.

To prove Theorem A, we introduce some subsets in $\mathrm{ClDiff}(M)$ as follows.

$T(M) = \{f : f$ is strongly C^0 tolerance stable$\}$

$T_\epsilon(M) = \{f :$ there exists $\delta > 0$ such that, for every $g \in V_\delta(f)$, every f-orbit is ϵ-traced by some g-orbit and every g-orbit is ϵ-traced by some f-orbit$\}$.

Lemma 4. *For every* $\epsilon > 0$, $T_\epsilon(M)$ *includes an open and dense subset in* $\mathrm{ClDiff}(M)$.

Proof. By Lemma 3, $T(M)$ is dense in $\mathrm{ClDiff}(M)$. Therefore it is sufficient to show that every $h \in T(M)$ is an interior point of $T_\epsilon(M)$ for every $\epsilon > 0$.

Let us take every $h \in T(M)$ and $\epsilon > 0$. Then there exists $\delta > 0$ such that for every $k \in V_{2\delta}(h)$, every h-orbit is $\epsilon/2$-traced by some k-orbit and every k-orbit is $\epsilon/2$-traced by some h-orbit. If $f \in V_\delta(h)$ and $g \in V_\delta(f)$, then f, $g \in V_{2\delta}(h)$. Therefore, if $f \in V_\delta(h)$, then $f \in T_\epsilon(M)$.

By Lemma 4 and the equality below, Theorem A (0) is proved.

$$T(M) = \bigcap_{n\geq 1} T_{1/n}(M).$$

Moreover, by Proposition 1, (1) and (2) are valid.

By Shub's density theorem, a topologically stable $f \in \mathrm{ClDiff}(M)$ has finite chain components, see [1]. Therefore, by Lemma 5 below, Theorem A (3) is proved. We can prove Lemma 5 by the same method of the proof of [5, Theorem 1 (h)].

Lemma 5. *A generic* $f \in \mathrm{ClDiff}(M)$ *has infinite chain components.*

Acknowledgment

The author wishes to thank Professor Kenichi Shiraiwa for criticism and encouragement to form the paper.

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