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A NOTE ON HENRICI'S TRIPLE PRODUCT THEOREM

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ABSTRACT. Making use of certain known transformations in the theory of hypergeometric functions, the authors prove a general triple series identity which readily yields Henrici's recent result expressing the product of three hypergeometric ${}_0F_1$ functions in terms of a hypergeometric ${}_2F_7$ function.

Recently, Henrici [2] derived the elegant formula:

$$(1)\quad {}_0F_1\left[\begin{matrix} -; \\ 6c; \end{matrix}\; x\right] {}_0F_1\left[\begin{matrix} -; \\ 6c; \end{matrix}\; \omega x\right] {}_0F_1\left[\begin{matrix} -; \\ 6c; \end{matrix}\; \omega^2 x\right]$$
$$= {}_2F_7\left[\begin{matrix} 3c-\frac{1}{4},\ 3c+\frac{1}{4}; \\ 6c,\ 2c,\ 2c+\frac{1}{3},\ 2c+\frac{2}{3},\ 4c-\frac{1}{3},\ 4c,\ 4c+\frac{1}{3}; \end{matrix}\; \left(\tfrac{4}{9}x\right)^3\right],$$

where

$$(2)\qquad \omega = \exp\left(\tfrac{2\pi i}{3}\right),$$

by consideration of the differential equations satisfied by the functions on either side of (1). In the present note we give a shorter proof of (1), utilizing certain known transformations of hypergeometric functions. Indeed, we shall first prove the following general triple series identity:

$$(3)\quad \sum_{m,n,p=0}^{\infty} \Delta_{m+n+p} \frac{\omega^{n+2p}}{(b)_m (b)_n (b)_p} \frac{x^{m+n+p}}{m!\,n!\,p!}$$
$$= \sum_{r=0}^{\infty} \frac{\Delta_{3r} \left(\frac{1}{2}b-\frac{1}{4}\right)_r \left(\frac{1}{2}b+\frac{1}{4}\right)_r}{(b)_r \left(\frac{1}{3}b\right)_r \left(\frac{1}{3}b+\frac{1}{3}\right)_r \left(\frac{1}{3}b+\frac{2}{3}\right)_r \left(\frac{2}{3}b-\frac{1}{3}\right)_r \left(\frac{2}{3}b\right)_r \left(\frac{2}{3}b+\frac{1}{3}\right)_r} \frac{\left(\frac{4}{9}x\right)^{3r}}{r!},$$

where $\{\Delta_n\}_{n=0}^{\infty}$ is a bounded sequence of complex numbers, and ω is defined by (2). For $b = 6c$ and $\Delta_n = 1$ $(n \geq 0)$, the identity (3) evidently yields Henrici's formula (1).

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Let Ω denote the left-hand side of (3). By employing some elementary series manipulations, it is not difficult to show that

$$\Omega = \sum_{k=0}^{\infty} \frac{\Delta_k \Lambda_k}{(b)_k} \frac{x^k}{k!}, \tag{4}$$

where

$$\Lambda_k = F_4\left[-k, 1-b-k; b, b; \omega, \omega^2\right] \tag{5}$$

in terms of the Appell F_4 function defined by

$$F_4[\alpha, \beta; \gamma, \delta; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\delta)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad \left(\sqrt{|x|} + \sqrt{|y|} < 1\right). \tag{6}$$

With the aid of the well-known quadratic transformation (cf., e.g., [3, p. 303, Equation 9.4(97)]):

$$F_4\left[\alpha, \alpha + \tfrac{1}{2} - \beta; \gamma, \beta + \tfrac{1}{2}; x, y^2\right]$$
$$= (1+y)^{-2\alpha} F_2\left[\alpha, \alpha + \tfrac{1}{2} - \beta, \beta; \gamma, 2\beta; \frac{x}{(1+y)^2}, \frac{4y}{(1+y)^2}\right],$$

we find from (5) that

$$\Lambda_k = \omega^k F_2\left[-k, 1-b-k, b-\tfrac{1}{2}; b, 2b-1; 1, 4\right], \tag{7}$$

since $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$.

Now write this Appell F_2 function as a single series whose terms involve a terminating ${}_2F_1[1]$. Applying the Chu-Vandermonde theorem in order to sum this ${}_2F_1[1]$ series, we have

$$\Lambda_k = \frac{(4\omega)^k (b-\frac{1}{2})_k}{(2b-1)_k} \, {}_3F_2\left[\begin{matrix} -k, & b-\frac{1}{2}, & 1-b-k; \\ & 2b-1, & 2-2b-2k; \end{matrix} \; 4\right]. \tag{8}$$

For the polynomial on the right-hand side of (8), we shall now prove the summation formula:

$${}_3F_2\left[\begin{matrix} -k, & b-\frac{1}{2}, & 1-b-k; \\ & 2b-1, & 2-2b-2k; \end{matrix} \; 4\right] = \begin{cases} 0, & k \not\equiv 0 \pmod 3, \\ \dfrac{2^{-2r}(3r)!\left(b-\frac{1}{2}\right)_{2r}}{r!\,(b)_r\left(b-\frac{1}{2}\right)_{3r}}, & k = 3r \; (r \in \mathbb{N}_0), \end{cases} \tag{9}$$

which, when applied to (8), yields

$$\Lambda_{3r} = \frac{2^{4r}(3r)!\,(b-\frac{1}{2})_{2r}}{r!\,(b)_r(2b-1)_{3r}}, \qquad \Lambda_{3r+1} = 0, \quad \text{and} \quad \Lambda_{3r+2} = 0. \tag{10}$$

Upon substitution from (10) into (5), we observe that

$$\Omega = \sum_{r=0}^{\infty} A_r \Delta_{3r} x^{3r}, \tag{11}$$

where

$$A_r = \frac{\Lambda_{3r}}{(3r)!\,(b)_{3r}} = \frac{2^{4r}\left(b-\frac{1}{2}\right)_{2r}}{(b)_r\,(b)_{3r}\,(2b-1)_{3r}\,r!}$$

$$= \frac{2^{6r}\left(\frac{1}{2}b-\frac{1}{4}\right)_r\left(\frac{1}{2}b+\frac{1}{4}\right)_r}{3^{6r}\,(b)_r\left(\frac{1}{3}b\right)_r\left(\frac{1}{3}b+\frac{1}{3}\right)_r\left(\frac{1}{3}b+\frac{2}{3}\right)_r\left(\frac{2}{3}b-\frac{1}{3}\right)_r\left(\frac{2}{3}b\right)_r\left(\frac{2}{3}b+\frac{1}{3}\right)_r r!}.$$

This evidently completes the proof of the general triple series identity (3).

In order to prove the summation formula (9), we recall Bailey's cubic transformation [1, p. 249, Equation (4.06)]:

$${}_3F_2\left[\begin{matrix} 3\alpha,\ \beta,\ 3\alpha-\beta+\frac{1}{2}; & 12 \\ & 4x \\ 2\beta,\ 6\alpha-2\beta+1; & \end{matrix}\right]$$

$$= (1-x)^{-3\alpha}\,{}_3F_2\left[\begin{matrix} \alpha,\ \alpha+\frac{1}{3},\ \alpha+\frac{2}{3}; & \frac{27x^2}{4(1-x)^3} \\ \beta+\frac{1}{2},\ 3\alpha-\beta+1; & \end{matrix}\right],$$

with 3α equal to zero or a negative integer. The apparent singularity at $x = 1$ is dealt with by series reversal. First, let $\alpha = -r$, where $r \in \mathbb{N}_0$. The right-hand side of (12) may then be transformed as follows:

$$(1-x)^{3r}\,{}_3F_2\left[\begin{matrix} -r,\ -r+\frac{1}{3},\ -r+\frac{2}{3}; & \frac{27x^2}{4(1-x)^3} \\ \beta+\frac{1}{2},\ 1-3r-\beta; & \end{matrix}\right] \tag{13}$$

$$= \frac{(1-x)^{3r}\left(\frac{1}{3}-r\right)_r\left(\frac{2}{3}-r\right)_r}{\left(\beta+\frac{1}{2}\right)_r(1-\beta-3r)_r}\left[\frac{-27x^2}{4(1-x)^3}\right]^r$$

$$\times\,{}_3F_2\left[\begin{matrix} -r,\ \frac{1}{2}-r-\beta,\ 2r+\beta; & \frac{4(1-x)^3}{27x^2} \\ \frac{1}{3},\ \frac{2}{3}; & \end{matrix}\right]$$

$$= \frac{\left(\frac{1}{4}x^2\right)^r(3r)!\,(\beta)_{2r}}{r!\left(\beta+\frac{1}{2}\right)_r(\beta)_{3r}}\,{}_3F_2\left[\begin{matrix} -r,\ \frac{1}{2}-r-\beta,\ 2r+\beta; & \frac{4(1-x)^3}{27x^2} \\ \frac{1}{3},\ \frac{2}{3}; & \end{matrix}\right].$$

The last expression in (13) is based upon some elementary Pochhammer symbol manipulations. Clearly, we may now take $x = 1$, and this proves (9) in the case $k = 3r$. We next perform similar transformations of the right-hand side of (12) in the cases when $\alpha = -r - \frac{1}{3}$ and $\alpha = -r - \frac{2}{3}$, and we find that the exponents of $(1-x)$ do not cancel: we are left with factors $(1-x)$ and $(1-x)^2$, respectively. Thus, taking $x = 1$, we obtain zero in these cases, and (9) is completely established.

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