PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 110, Number 1, September 1990

A CLASS OF SHIFTS ON THE HYPERFINITE II_1 FACTOR

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We construct and classify up to conjugacy certain shifts on the hyperfinite II_1-factor, each being a shift of Jones index n which fails to be an n-shift. In particular for each prime n we construct uncountably many such shifts.

A shift σ on a von Neumann algebra R is an isomorphism of R into itself such that $\bigcap_{k=0}^{\infty} \sigma^k(R) = \mathbb{C}$. Shifts were introduced by R. T. Powers in [8] and studied further in [4], [9], and [10]. In [2] and [3] we introduced the notion of group shift, thereby generalizing and unifying many of the earlier results. An n-shift is the simplest kind of shift, coming from the group $\oplus_{k=0}^{\infty} Z_n^{(k)}$ with its canonical shift. The first example of a shift with index n which is not an n-shift was constructed in [9] for $n = 2$. Recently, in [5], Choda, Enomoto, and Watatani, using essentially the methods of [2], constructed uncountably many shifts of index 2 which are not 2-shifts. To construct a group shift, we need to specify a group G, a shift s on G, and a nondegenerate s-invariant 2-cocycle ω on G. We fix G to be $\oplus_{k=-\infty}^{\infty} Z_n^{(k)}$ and s to be given by $s(e_j) = e_j + e_{j+1}$ for all integers j, where e_j is a generator of the jth copy of Z_n. By [2], provided s is a shift on G and ω is nondegenerate, the data G, s, ω give rise to a shift $\sigma = \sigma(G, s, \omega)$ on the hyperfinite II_1-factor R realized as the twisted group von Neumann algebra $W^*(G, \omega)$.

In concrete terms, R is presented as generated by a family of unitaries $(u_k)_{k\in Z}$ of order n with commutation relations

$$u_i u_j u_i^* u_j^* = \omega(e_i, e_j)\overline{\omega(e_j, e_i)}.$$

The shift σ takes u_k to $\overline{\omega(e_k, e_{k+1})} u_k u_{k+1}$.

We note that it is easy to see that σ is a shift; the difficulty lies in proving that any particular cocycle is nondegenerate. From there it is relatively easy using [2] to see which cocycles give rise to conjugate shifts. Hence we choose a class of s-invariant cocycles which are relatively easy to show nondegenerate.

It would be possible to carry out the same kind of calculations for the shift $e_i \to e_i + e_{i+1} + \cdots + e_{i+m}$ for fixed m, but the details of specifying an invariant

Received by the editors September 6, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L10.

cocycle, and especially of proving nondegeneracy, would be tedious. We would expect that for distinct m we would obtain distinct classes of shifts (just as in this paper we prove that the cases $m = 1$ and n-shifts give nonconjugate shifts whatever the choice of cocycle).

We will now summarize our results and methods in a sequence of lemmas and propositions. The proofs follow at the end of the paper.

First, in order to apply the results of [2], we need to show that s is a shift and that $[\sigma(R)]' \cap R = \mathbb{C}$. According to [2], the latter condition is equivalent to $\rho(g \wedge s(G)) = 1$ implying $g = 0$, where ρ is the antisymmetric character of $G \wedge G$ associated with ω: $\rho(g \wedge h) = \omega(g, h)\overline{\omega(h, g)}$.

Lemma 1.

(i) *s is a shift on G.*

(ii) *For every nondegenerate s-invariant 2-cocycle ω on G, and for all $k = 0, 1, 2, \dots$, $\rho(g \wedge s^k(G)) = 1$ implies $g = 0$. Hence $[\sigma^k(R)]' \cap R = \mathbb{C}$ for $k = 0, 1, 2, \dots$.*

Now [2; Proposition 2.5] gives us a preliminary classification result (Proposition 2). We remark also that the derived tower of σ, $(\sigma^k(R)' \cap R)_{k=0,1,\dots}$, is trivial and therefore the methods of [3] provide no information on outer conjugacy of the σ's.

Propositon 2. *Suppose $\sigma_i = \sigma(G, S, \omega_i)$, $i = 1, 2$, are the group shifts of Lemma 1. Then σ_1 and σ_2 are conjugate if and only if there exists a group automorphism ϕ of G such that $\phi \circ s = s \circ \phi$ and $[\omega_1] = [\omega_2 \circ \phi]$ in the second cohomology group $H^2(G; T)$ where T is the unit circle.*

To apply Proposition 2 we need to determine which ϕ in $\operatorname{Aut}(G)$ commute with s. Let $\gamma \in \operatorname{Aut}(G)$ be defined by $\gamma(e_j) = e_{j+1}$ for all integers j. Since $s = \gamma + \text{identity}$, ϕ commutes with s if and only if ϕ commutes with γ.

Lemma 3. (i) *Those $\phi \in \operatorname{Aut}(G)$ which commute with s are in one-to-one correspondence with those $\rho \in G$ such that $\{\gamma^k(\rho): k \in Z\}$ generates G, via $\rho = \phi(e_0)$.*

(ii) *Provided n is a prime, such a g is of the form $g = me_t$ for some nonzero $m \in Z_n$ and some $t \in Z$. Thus $\phi \in \operatorname{Aut}(G)$ commutes with s if and only if there exist a nonzero $m \in Z_n$ and $t \in Z$ such that $\phi(e_k) = me_{k+t}$.*

We remark that when n is not prime there are other such g. For example, let $n = 4$ and $g = 2e_0 + e_1 + 2e_2$. Then $\gamma^{-1}(2g) + g + \gamma(2g) = e_1$.

We now specify ω. An s-invariant 2-cocyle ω on G, or equivalently an s-variant antisymmetric character ρ of the exterior product $G \wedge G$, can be described by a doubly infinite matrix $(a_{j,k})$ where the $a_{j,k} \in Z_n$ are defined by

$$\rho(e_j \wedge e_k) = \exp[(2\pi i/n)a_{j,k}] \tag{1}$$

and necessarily satisfy

(2) $$a_{j,j} = 0 \quad \text{and} \quad a_{j,k} + a_{k,j} = 0 \qquad \text{for all } j, k \in Z.$$

Now the s-invariance condition is

(3) $$a_{j,k+1} + a_{j+1,k+1} + a_{j+1,k} = 0 \qquad \text{for all } j, k \in Z.$$

We say that the group-shift $\sigma = \sigma(G, s, \omega)$ is associated with the matrix $(a_{j,k})$.

Proposition 4. *Suppose σ_1 and σ_2 are the group-shifts associated with the matrices $(a_{j,k})$ and $(b_{j,k})$, respectively, and suppose that n is prime. Then σ_1 and σ_2 are conjugate if and only if there exist integers $m \neq 0$ in Z_n and t such that $a_{j,k} = m^2 b_{j+t,k+t}$ for all $j, k \in Z$.*

We now specialize to a class of ω's which are nondegenerate. To specify a matrix $(a_{i,j})$ satisfying (2) and (3) it is sufficient to specify one row, say $(a_{0,j})_{j\in Z}$. We specialize to

$$a_{0,j} = \begin{cases} 0 & \text{for } j \leq 0 \\ 1 & \text{for } j = 1 \\ x_j & \text{for } j \geq 2 \end{cases}$$

so that the sequence $x_2, x_3, \ldots$ of elements of Z_n determines ω and hence σ.

Lemma 5. *Every sequence $x_2, x_3, \ldots$ of elements of Z_n determines (through (a_{jk})) a nondegenerate s-invariant 2-cocycle ω of G, and hence a shift $\sigma = \sigma(G, s, \omega)$ on the hyperfinite II_1-factor R.*

Proposition 6. *Let $\sigma_i = \sigma(G, s, \omega_i)$ be the group shifts determined by $(x_k)_{k=2,3,\ldots}$ and $(y_k)_{k=2,3,\ldots}$ as described above, and let n be prime. Then σ_1 and σ_2 are conjugate if and only if $x_k = y_k$ for all $k \geq 2$.*

Corollary 7. *For every prime n, there are uncountably many mutually nonconjugate group shifts on the hyperfinite II_1-factor R, each being of index n and failing to be an n-shift.*

There is another shift on $G = \oplus_{j=-\infty}^{\infty} Z_n^{(j)}$, namely t defined by

$$t(e_j) = \begin{cases} e_{j+1} & \text{for } j \geq 0 \\ e_j + e_{j+1} & \text{for } j < 0. \end{cases}$$

If ω is a nondegenerate t-invariant 2-cocyle on G we can consider the group-shift $\sigma(G, t, \omega)$. When $n = 2$ the shifts $\sigma(G, t, \omega)$ are essentially those constructed by Price in [9; §5]. We remark that in fact the class of shifts $\sigma(G, t, \omega)$ and the $\sigma(G, s, \omega)$ discussed above coincide, so that our classification results apply to the shifts of [9; §5]. This follows from the fact that s and t are conjugate: there exists a ϕ in $\mathrm{Aut}(G)$ such that $\phi \circ t = s \circ \phi$. Specifically, define a homomorphism $\phi: G \to G$ by

$$\phi(e_j) = \begin{cases} s^j(e_0) & \text{for } j \geq 1 \\ e_j & \text{for } j \leq 0. \end{cases}$$

Notice that $\phi(e_j) = \sum_{k=0}^{j} \binom{j}{k} e_k$ for $j \geq 0$ so ϕ is in fact an automorphism of G. A direct calculation confirms that $\phi \circ t = s \circ \phi$. Notice that s being a shift implies t is a shift.

Proof of Lemma 1. (i) To see that s is a shift, note that every nonzero $g \in G$ can be written uniquely in the form $g = \sum_{j=m_1}^{m_2} a_j e_j$ where $a_j \in Z_n$ and $a_{m_1} \neq 0$, $a_{m_2} \neq 0$. Define $l(g) = m_2 - m_1$. Evidently $l(s(g)) = l(g) + 1$ for all $g \in G - \{0\}$. Hence if g is in $s^k(G) - \{0\}$, $l(g) \geq k$. Therefore $\bigcap_{k=0}^{\infty} s^k(G) = \{0\}$ so that s is a shift.

(ii) Suppose that ω is an s-invariant nondegenerate 2-cocycle on G and that ρ is the associated antisymmetric bicharacter. Let γ in $\operatorname{Aut}(G)$ be defined by $\gamma(e_j) = e_{j+1}$ for all $j \in Z$ so that $s = \gamma + i$ where i is the identity. From the s-invariance we obtain:

$$\begin{aligned} \rho(g \wedge h) &= \rho(s(g) \wedge s(h)) \\ &= \rho((\gamma(g) + g) \wedge s(h)) \\ &= \rho(\gamma(g) \wedge s(h))\rho(g \wedge (\gamma(h) + h)) \\ &= \rho(\gamma(g) \wedge s(h))\rho(g \wedge \gamma(h))\rho(g \wedge h). \end{aligned}$$

Hence $\rho(\gamma(g) \wedge s(h)) = (\rho(g \wedge \gamma(h)))^{-1}$. Replacing g by $\gamma^{-1}(g)$ in the above, we get:

$$\rho(g \wedge s(h)) = (\rho(\gamma^{-1}(g) \wedge \gamma(h)))^{-1}. \tag{4}$$

Now assume that $\rho(g \wedge s^k(G)) = 1$. Then by (4), $\rho(\gamma^{-1}(g) \wedge \gamma(s^{k-1}(G))) = 1$, or, since γ and s commute, $\rho(\gamma^{-1}(g) \wedge s^{k-1}(G)) = 1$. By induction it follows that $\rho(\gamma^{-k}(g) \wedge G) = 1$; from there, since ω is nondegenerate, $\gamma^{-k}(g) = 0$ and $g = 0$.

By [2; Corollary 1.3] this implies that $(\sigma^k(R))' \cap R = \mathbb{C}$.

Proof of Proposition 2. By [2; Proposition 2.5] σ_1 and σ_2 are conjugate if and only if there exist $\phi \in \operatorname{Aut}(G)$ and a map $\lambda: G \to T$ such that:

(i) $\phi \circ s = s \circ \phi$,
(ii) $\omega_1(g, h) = \lambda(g)\lambda(h)(\lambda(gh))^{-1}\omega_2(\phi(g), \phi(h))$ for all $g, h \in G$, and
(iii) $\lambda(s(g)) = \lambda(g)$ for all $g \in G$.

Therefore σ_1 and σ_2 conjugate evidently implies there exists $\phi \in \operatorname{Aut}(G)$ with $\phi \circ s = s \circ \phi$ and $[\omega_1] = [\omega_2 \circ \phi]$. Conversely suppose such a ϕ exists. Then there exists $\lambda: G \to T$ such that (i) and (ii) above hold. Define a map $\psi: G \to T$ by $\psi(g) = \lambda(g)\overline{\lambda(s(g))}$; then ψ is a group homomorphism by (ii) since ω_1 and ω_2 are s-invariant. Then define a group homomorphism $\theta: G \to T$ by

$\theta(e_j) = \psi(e_{j-1})$ for all j. It is straightforward to check that $\theta(s(g))\lambda(s(g)) = \theta(g)\lambda(g)$ for all $g \in G$. Hence $\theta\lambda: g \to \theta(g)\lambda(g)$ satisfies (ii) and (iii) above, from which it follows that σ_1 and σ_2 are conjugate.

Proof of Lemma 3. (i) (Compare the classical computation of the commutant of the bilateral shift, e.g. [1, Theorem 2].) Assume that ϕ is an automorphism of G which commutes with s. Then ϕ commutes with γ. Let $g = \phi(e_0)$. Then $\gamma^k(g) = \phi(\gamma^k(e_0)) = \phi(e_k)$ so $\{\gamma^k(g): k \in Z\}$ generates G, and g completely determines ϕ.

Conversely suppose that $\{\gamma^k(g): k \in Z\}$ generates G and define a homomorphism $\phi: G \to G$ by $\phi(e_k) = \gamma^k(g)$. Then ϕ commutes with γ and hence s, and ϕ is surjective. But ϕ is also one-to-one because $\{\gamma^k(g): k \in Z\}$ is Z_n-linearly independent. This is evident by a length argument when n is prime; to see it in general, assume we have a relation $\sum_{j=m}^{m+l} c_j\gamma^j(g) = 0$. Choose k so that $H = \oplus_{j=k}^{\infty} Z_n^{(k)}$ contains all the $\gamma^j(g)$ for $j \geq m$. Then γ is a shift on H, and the argument used in the proof of [2; Proposition 4.1] shows that $c_j = 0$ for all j.

(ii) Suppose that n is prime and that $\{\gamma^k(g): k \in Z\}$ generates G. Write $g = \sum_{j=j_0}^{j_0+l} x_j e_j$ where $x_{j_0} \neq 0$ and $x_{j_0+l} \neq 0$, so that $l(g) = l$ (see the proof of Lemma 1). Suppose that h is of the form $h = \sum_{k=k_0}^{k_0+m} y_k\gamma^k(g)$ with $y_{k_0} \neq 0$ and $y_{k_0+m} \neq 0$. Because n is prime, $x_{j_0}y_{k_0} \neq 0$ and $x_{j_0+l}y_{k_0+m} \neq 0$; therefore $l(h) = l + m$. In particular $l(h) \geq l(g)$, so that the assumption that $\{\gamma^k(g): k \in Z\}$ generates G implies $l(g) = 0$. That means $g = me_t$ with $m \neq 0$. The corresponding ϕ with $\phi(e_0) = g$ satisfies $\phi(e_k) = \gamma^k\phi(e_0) = me_{k+t}$.

Proof of Proposition 4. Suppose σ_1 and σ_2 are conjugate. Then, by Proposition 2, there exists $\phi \in \mathrm{Aut}(G)$ such that $\phi \circ s = s \circ \phi$ and $\rho_1 = \rho_2 \circ \phi$. By Lemma 3 there exist integers $m \neq 0$ and t such that $\phi(e_k) = me_{k+t}$. It follows that

$$\rho_1(e_j \wedge e_k) = \rho_2(\phi(e_j) \wedge \phi(e_k)) = m^2\rho_2(e_{j+t}, e_{k+t})$$

so that $a_{j,k} = m^2 b_{j+t,k+t}$ for all $j, k \in Z$.

Conversely, if $a_{j,k} = m^2 b_{j+t,k+t}$ for all $j, k \in Z$, define $\phi \in \mathrm{Aut}(G)$ by $\phi(e_k) = me_{k+t}$. Then $\phi \circ s = s \circ \phi$ and $\rho_1 = \rho_2 \circ \phi$, and σ_1 and σ_2 are conjugate.

Proof of Lemma 5. Assume $a_{0,j} = 0$ for $j \leq 0$ and $a_{0,1} = 1$. Let A_k be the $2k \times 2k$ matrix $(a_{i,j})_{-k+1 \leq i,j \leq k}$. To show that ω is nondegenerate, it suffices to show that each A_k is nonsingular (cf. [2, §5]). From (2) and (3) we easily see that $a_{i,j} = 0$ for $i, j \leq 0$ and that

$$A = \begin{bmatrix} 0 & B_k \\ -B_k^T & * \end{bmatrix}$$

where B_k is the $k \times k$ matrix of $a_{i,j}$'s with $-k+1 \leq i \leq 0$ and $1 \leq j \leq k$. It thus suffices to show that each B_k is nonsingular. Denote by R_i the infinite row $(a_{i,j})_{j \in Z}$. Note that $R_0 = (\dots, 0, 0, 1, x_2, x_3, \dots)$. Condition (3) can be rewritten as $R_i + R_{i+1} = -tR_{i+1}$ where t denotes the translation to the right by one step. Then $R_0 + R_{-1} = -tR_0$, $R_0 + 2R_{-1} + R_{-2} = -tR_0 - tR_{-1} = t^2 R_0$, and in general

$$\sum_{j=0}^{m} \binom{j}{m} R_{-j} = (-1)^j t^j R_0.$$

It follows that B_k can be reduced by row operations to a lower triangular matrix with 1's on the diagonal and is therefore nonsingular. We have shown that ω is nonsingular.

Proof of Proposition 6. Assume $(x_k)_{k=2,3,\dots}$ and $(y_k)_{k=2,3,\dots}$ determine matrices $(a_{i,j})$ and $(b_{i,j})$, respectively, and assume σ_1 and σ_2 are conjugate. By Proposition 4, there exist integers $m \neq 0$ and t such that $a_{i,j} = m^2 b_{i+t,j+t}$ for all i, j. Since the upper-left corners of $(a_{i,j})$ and $(b_{i,j})$, for $i \leq 0$, $j \leq 0$, are identically zero and are bounded by

$$\begin{array}{cccc|c} & & & & \vdots \\ & & & & +1 \\ & & 0 & & -1 \\ & & & & +1 \\ \hline \cdots & -1 & +1 & -1 & 0 \end{array}$$

the only possibility is $t = 0$. Hence $a_{i,j} = m^2 b_{i,j}$ for all i, j. In particular $a_{0,1} = 1$, $b_{0,1} = 1$ implies $m^2 = 1$ and $a_{i,j} = b_{i,j}$ for all i, j. Hence $x_k = a_{0,k} = b_{0,k} = y_k$. The converse is evident.

Proof of Corollary 7. This follows from Proposition 6 and [2; Proposition 5.2].

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