PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 110, Number 1, September 1990

AN INFINITE DIMENSIONAL EXTENSION OF THEOREMS OF HARTMAN AND WINTNER ON MONOTONE POSITIVE SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

A. F. IZÉ

(Communicated by Kenneth R. Meyer)

Dedicated to Jack K. Hale on his sixtieth anniversary

ABSTRACT. Consider the equation (1) $\dot{x} + A(t)x = -f(t, x)$ $x(0) = x^0$, $x^0 \in X$, a Banach sequence space with a Schauder Basis. It is proved that if $f(t, 0) = 0$, $A(t)(\cdot) + f(t, \cdot)$ is a positive operator and the solution operator $K(t, 0)x^0 = x^0 - \int_0^t A(s)\,ds - \int_0^t f(s, x(s))\,ds$ is compact for $t > 0$, then system (1) has at least one solution $x(t)$, $x(t) \not\equiv 0$ such that $x(t) \geq 0$, $-\dot{x}(t) \leq 0$, and consequently $x(t)$ are monotone nonincreasing for $t \geq 0$.

1. INTRODUCTION

In [5] we presented a topological method that can be applied to study the asymptotic behavior of differential equations in Banach spaces. The book of Krasnoselskii [8] is dedicated to the determination of positive solutions of operator equations. If X is a Banach space with a cone C, the linear operator A is called positive if it transforms the cone into itself. It follows from $x \geq 0$ that $Ax \geq 0$.

The operator A is strongly positive if $x \geq 0$, $x \neq 0$, implies $Ax > 0$. (This definition applies also when the cone is not solid). The operator A is negative if it transforms the cone C into $-C$. It follows from $x \geq 0$ that $-Ax \geq 0$. A is strongly negative if $x \geq 0$, $x \neq 0$, implies $-Ax \geq 0$. Even when the operator $A\colon R^+ \times X \to X$ is negative, the differential equation $\dot{x} = A(t, x)$, $A(t, 0) = 0$ may have positive solutions. In this paper we apply the results proved in [5] to extend theorems of Hartman and Wintner [2] and [3], to an infinite dimensional space. The proof shows the generality and simplicity of the methoddeveloped in [5].

Received by the editors January 4, 1989 and, in revised form, September 30, 1989; this paper has been presented at the International Conference on Theory and Applications of Differential Equations, Ohio University, Columbus, Ohio, March 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34C35, 34K30, 58D25.

Key words and phrases. Differential equations, Banach spaces, infinite dimensional spaces, positive solutions, operator equations, strongly positive, solid cone, egress points, strict egress points, trajectory, orbit, consequent operator, left shadow, process, retract, infinitesimal generator, classical solution.

0002-9939/90 $1.00 + $.25 per page

2. Preliminaries

We begin by recalling a few definitions and results from [5].

Suppose X is a Banach space, $R^+ = [0, \infty)$, $u: R \times X \times R^+ \to X$ is a given mapping and define $U(\sigma, t): X \to X$ for $\sigma \in R$, $t \in R^+$ by $U(\sigma, t)x = u(\sigma, x, t)$. A process on X is a mapping $u: R \times X \times R^+ \to X$ satisfying the following properties:

(i) u is continuous,

(ii) $U(\sigma, \sigma) = I$ (identity),

(iii) $U(\sigma + s, t)U(\sigma, s) = U(\sigma, s + t)$.

A process is said to be an autonomous process or a semidynamical system if $U(\sigma, t)$ is independent of σ, i.e., $U(\sigma, t) = U(0, t)$ for each $\sigma \in R$ and $t \geq 0$. When this is the case, define $T(t) = U(0, t)$ and note $T(t)x$ is continuous at each $(t, x) \in R \times X$.

Definition. Suppose u is a process on X. The trajectory $\tau^+(\sigma, x)$ through $(\sigma, x) \in R \times X$ is the set in $R \times X$ defined by

$$\tau^+(\sigma, x) = \{(\sigma + t, U(\sigma, t)x) \mid t \in R^+\}.$$

The orbit $\gamma^+(\sigma, x)$ through (σ, x) is the set in X defined by

$$\gamma^+(\sigma, x) = \{U(\sigma, t)x \mid t \in R^+\}.$$

An integral of the process on R is a continuous function $y: R \to X$ such that for any $\sigma \in R$, $\tau^+(\sigma, y(\sigma)) = \{(\sigma + t, y(\sigma + t)) | t \geq 0\}$. An integral y is an integral through $(\sigma, x) \in R \times X$, if $y(\sigma) = x$.

We assume in the following that the integral through each $(\sigma, x) \in R \times X$ is unique. We define $\tau^{-1}(x) = \{(\sigma, y) \in R \times X | \exists\ t > 0$ such that $U(\sigma, t)y = x\}$. If $P_0 = (\sigma, x) \in R \times X$ and $z \in \gamma^+(\sigma, x)$, we define

$$t_z = \inf\{t \geq 0 \mid U(\sigma, t)x = z\},$$
$$Q_z = (\sigma + t_z, U(\sigma, t_z)x),$$
$$[P_0, Q_z] = (\sigma + t, U(\sigma, t)x \mid 0 \leq t \leq t_z\}.$$

Let Ω be an open set of $R \times X$, ω an open set of Ω, $\omega \neq \phi$ and $\partial\omega = \overline{\omega} \cap (\overline{\Omega - \omega})$ the boundary of ω with respect to Ω. We put

$$S^0 = \{P_0 = (\sigma, x) \in \partial\omega \mid \exists z \in \gamma^+(\sigma, x) \text{ with} (P_0, Q_z) \neq \phi \text{ and } (P_0, Q_z) \cap \overline{\omega} = \phi\}$$

$$S = \{Q \in \partial\omega \mid \exists P_0 = (\sigma, x) \in \omega \text{ with } Q \in \tau^+(\sigma, x) \text{ and } [P_0, Q) \subset \omega\},$$

$$S^* = S^0 \cap S.$$

The points of S are called egress points. The points of S^* are called strict egress points.

Given a point $P_0 = (\sigma, x) \in \omega$, if the trajectory $\tau^+(\sigma, x)$ of the process is contained in ω for every $t > 0$ we say that the trajectory is asymptotic with respect to ω. If the trajectory is not asymptotic with respect to ω then there is a $t > 0$ such that $(\sigma + t, U(\sigma, t)x) \in \partial\omega$. Taking:

$$t_{p_0} = \min\{t > 0 \mid (\sigma + t, U(\sigma, t)x) \in \partial\omega\}$$
$$Q = (\sigma + t_{p_0}, U(\sigma, t_{p_0})x) = C(P_0)$$

we have

$$[P_0, Q] \subset \omega.$$

The point $C(P_0)$ is called the consequent of P_0. Define G to be the set of all $P_0 = (\sigma, x) \in \omega$ such that there are $C(P_0)$ and $C(P_0) \in S^*$. G is called the left shadow of ω. Consider the mapping, the consequent operator:

$$K: S^* \cup G \to S^*$$

$K(P_0) = C(P_0)$ if $P_0 \in \omega$, and $K(P_0) = P_0$ if $P_0 \in S^*$.

Lemma 1. *The consequent operator $K: S^* \cup G \to S^*$ is continuous. The following theorem is proved in* [5].

Theorem 1. *Let ω be a nonempty, open subset of Ω, and with S and S^* denoting respectively, the set of egress and strict egress points of ω, assume $\phi \neq Z \subset \omega \cup S$ and the following conditions are satisfied*:

(i) $S = S^*$,
(ii) *Z is a closed, bounded, convex set and K is a compact operator,*
(iii) *$Z \cap S$ is a retract of S, that is, there exists a retraction $r: S \to Z \cap S$.*
(iv) *there exists a continuous mapping $\Phi: Z \cap S \to Z \cap S$ such that $\Phi(P) \neq P$ for every $P \in Z \cap S$.*

Then there exists at least one point $P = (\sigma, x) \in Z - S$ such that the trajectory $\tau^+(\sigma, x)$ through $P = (\sigma, x)$ is contained in $\omega - S$.

3. Main results

Let X be a real Banach space with Schauder basis $\{e^i\}$ and identify each $x \in X$ with the coordinate sequence $(x_1, x_2, \cdots)$. For example, take $X = c$, the space of real convergent sequences with norm given by $\|x\| = \sup |x_i|$ where $x = \{x_i\}_{i=1}^\infty$ and for $i = 1, 2, \cdots$, let e^i be the natural basis, i.e., $\{e^i\}$ is the sequence whose ith element is 1 and has all other elements equal to zero.

The main results of this paper are for certain linear differential equations on the sequence space isomorphic to a Banach space X with a Schauder basis. The reader unfamiliar with the notion of sequence space is referred to the first three pages of [6] which contain sufficient material to understand sequence spaces as they are used here.

Consider the system

$$\dot{x}_i + \sum_{j=1}^{\infty} a_{ij}(t)x_j = 0 \qquad i = 1, 2, \cdots \tag{1}$$
$$x_i(0) = x_i^0$$

where $a_{ij}(t)$ are continuous functions of the real variable t for $0 \leq t < \infty$, $x_0 = (x_1^0, x_2^0, x_3^0, \cdots) \in X$ sequence space with a Schauder basis.

System (1) can be written in the form

$$\dot{x} + A(t)x = 0, \qquad x(0) = x^0. \tag{2}$$

We assume that for each $t \in [0, T]$, $-A(t)$ is the infinitesimal generator of a C^0 semigroup on the space X, the domain $D(A(t)) = D$ is independent of t, is dense in X and that the initial value problem (1) has a unique classical solution defined in $[0, \infty]$. Assume also continuity with respect to initial condition for the solutions of (1). See [4] and [7].

Our purpose here is to apply Theorem 1 to prove existence of a positive solution of system (1).

Theorem 2. *Assume the hypotheses*

(i) *The solution operator* $K(t, 0)x^0 = x^0 - \int_0^t A(s)x(s)\,ds$ *is compact for* $t > 0$;

(ii) $\sum_{j=1}^{\infty} a_{ij}(t)x_j \geq 0$ *for every* $i = 1, 2, \cdots, x_j \geq 0, \qquad j = 1, 2, \cdots$.

Then system (1) *has a monotone nonincreasing solution* $x(t) = (x_1(t), x_2(t)), \cdots)$, $x(t) \neq 0$ *such that* $x_i(t) \geq 0$ *and* $-\dot{x}_1(t) \geq 0$ *for every* $i = 1, 2, \cdots, t \geq 0$, *and consequently* $x_i(t)$ *are monotone nonincreasing for* $t \geq 0$.

Proof. Let X be a Banach space with Schauder basis $\{e^i\}_{i=1}^{\infty}$ and without loss of generality assume the basis is normalized so that $\|e^i\| = 1$ for each i. Identify $x \in X$ with the ∞-tuple $(x_1, x_2, \cdots)$ where $x = \sum_1^{\infty} x_i e^i$ is the unique expansion of x in terms of the basis $\{e^i\}_{i=1}^{\infty}$. Let us assume first that

$$\sum_{j=1}^{\infty} a_{ij}(t)x_j > 0, \qquad i = 1, 2, \cdots$$

then define

$$\omega = \{x \in X : x_i > 0 \quad \text{for } i = 1, 2, \cdots\}$$

$$S = \{x \in X : x_i \geq 0 \text{ for } i = 1, 2, \cdots \text{ and } x_j = 0 \text{ and } x_k \neq 0 \text{ for at least one } j \text{ and one } k\}$$

$$Z = \left\{x \in X : x_i \geq 0 \text{ and } \sum_1^{\infty} x_i = \delta, \qquad \delta > 0\right\}$$

Note that ω is open in X and ω is a solid cone in X. (It is the positive orthant relative to the Schauder basis.) Further it is easily verified that $S \cup \{0\}$ equals the boundary of $\overline{\omega}$ and $Z \subset \omega \cup S$. Z is a cross-section of the cone $\overline{\omega}$. Also Z is bounded since $x \in Z$ implies $\|x\| \leq \sum x_i \|e^i\| = \delta$.

In dimension 3 the sets ω, S and Z are shown in Figure 1. ω is the interior of the positive orthant, $S \cup \{0\}$ is the surface of the positive orthant and Z is the face of the 3-simplex with vertices $0, e^1, e^2, e^3$ opposite 0. If the space X

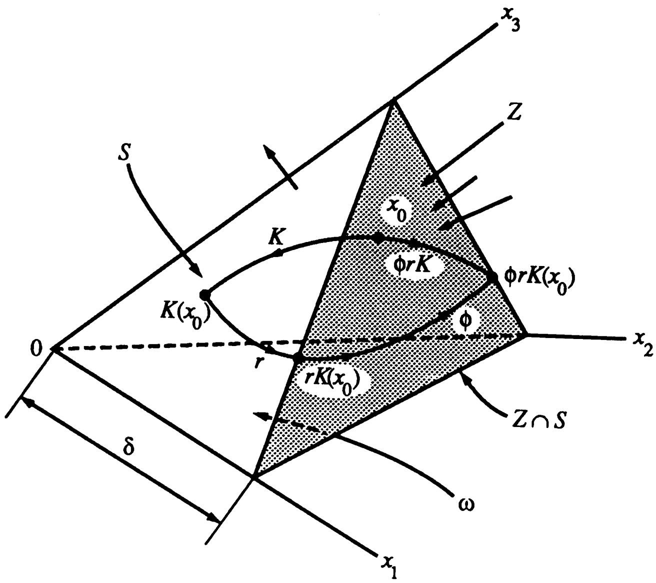

FIGURE 1.

is finite dimensional the proof has to be adapted defining $\omega = \{x \in X, : x_i > 0$ for $i = 1, 2, \cdots, n\}$ and $Z = \{x \in X; x_i \geq 0$ and $\sum_1^n x_i = 1\}$.

From hypothesis (3) $\dot{x}_i < 0$, then the derivatives along the solutions of (1) on the faces of the infinitehedron are negative, and the points of S are strict egress points. At the origin $\dot{x}_i = 0$ for every i, whence the origin is not an egress point. The boundary of Z is $\partial Z = x = \{(x_1, x_2, \cdots) | \sum_{i=1}^{\infty} x_i = \delta$, $x_i = 0$ for at least one $i\}$ if $x \in S$ and $\sigma(x) = \sum_1^{\infty} x_i$, then $\sigma: S \to R|\{0\}$ is continuous, and if we define

$$r(x) = \frac{\delta}{\sigma(x)} x \quad \text{for } x \in S$$

then $r: S \to Z \cap S = \partial Z$ is a retraction of S into $Z \cap S$. The continuous function $\Phi: Z \cap S \to Z \cap S$ defined by $\phi(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$ satisfies $\Phi(x) \neq x$ for every $x \in Z \cap S$. Assume that for any $x^0 \in Z \cap \omega$, the solution through x^0 does not remain in ω for $t > 0$, then $Z \cap \omega \subset G$.

The consequent operator $K: Z \to S$ defined by

$$(4) \qquad K(t, 0)(x^0)_i = x_i^0 - \int_0^{\bar{t}} \sum_{j=1}^{\infty} a_{ij}(s) x_j(s)\, ds, \qquad i = 1, 2, \cdots$$

is defined in $Z \cap \omega$. Since the points of S are strict egress, K is defined in $Z \cap S$ then in $\overline{Z}$, that is, $\overline{Z} \subset G \cup S$. From Lemma 1, K is continuous and from hypothesis i, K is compact. Since Φ and r are continuous functions, $\Phi \circ r \circ K$ is compact. From Theorem 1, there exists at least a $\overline{x} \in Z \cap \omega$ such that the solution of (1) $x(t) = (x_1(t), x_2(t) \cdots)$ through $\overline{x}$ stays in ω for every $t \geq 0$ and since $\dot{x}_1 < 0$, $i = 1, 2, \cdots$, each $x_i(t)$ decreases monotonically to zero as $t \to \infty$.

If

$$\sum_{j=1} a_{ij}(t)x_j \geq 0$$

Consider the system

$$\text{(5)} \qquad \begin{aligned} &\dot{y}_i + \sum_{j=1}(a_{ij}(t) + \varepsilon_{ij})y_j = 0 \\ &y_i(0) = y_i^0, \qquad i = 1, 2, \ldots \end{aligned}$$

$\varepsilon_{ij} > 0$ arbitrarily small.

From the proof above there exists for each ε_{ij} a positive solution of (5) through some $y_n^0 = (y_{1n}^0, y_{2n}^0, \cdots)$. When $\varepsilon_{ij} \to 0$ there is a sequence of positive solutions $y_n(t)$ of (5) through y_n^0. Let $E = \{y_n^0\}$. For $\bar{t} > t_0$, the set $\{K(\bar{t}, 0)(y_{in}^0)(\bar{t})\} = \{y_{in}(\bar{t})\}$ is compact and there exists a convergent subsequence $\{y_{in_k}(\bar{t})\}$, $y_{in_k}(\bar{t}) \to y_0(\bar{t})$ and the solutions $y_{in_k}(t) \to y(t)$, $y(\bar{t}) = y_0(\bar{t})$ on every interval $\bar{t} \leq t \leq T < \infty$.

Remark 1. In finite dimension, $X = R^n$, the operator K is compact and Theorem 2 is true without hypothesis (i). This is a Theorem of Hartman and Wintner [2].

Remark 2. Let $A = A(t)$ be a constant $n \times n$ matrix. Theorem 1 can be considered as a generalization of the algebraic theorem of Perron, which states that a nonnegative (constant) matrix A possesses at least one nonnegative eigenvalue λ corresponding to which there is a nonnegative eigenvector c. (If $A > 0$ then $\lambda > 0$ and $c > 0$.) (See Remark 2 in [2].)

The generalization of this result to infinite dimensional Banach sequence spaces, the Krein-Rutman Theorem, is also a consequence of Theorem 1.1.

The system

$$\text{(6)} \qquad \begin{aligned} &\dot{x}_i + \Sigma a_{ij}(t)x_j = -f_i(t, x) \\ &x_i(0) = x_i^0, \qquad i = 1, 2, \cdots \end{aligned}$$

can be written in the form

$$\text{(7)} \qquad \begin{aligned} &\dot{x} + A(t)x = -f(t, x) \\ &x(0) = x^0 \end{aligned}$$

We assume that for each $t \in [0, T]$, $-A(t)$ is the infinitesimal generator of a C^0-semigroup on the space X, the domain $D(A(t)) = D$ is independent of t, is dense in X, $f\colon [0, \infty) \times U \to X$ is continuous, $U \subset X$, open, $f(t, 0) = 0$ and we assume existence of a unique classical solution of (6) in $[0, \infty)$, as well as continuity with respect to initial conditions; see [4] and [7].

Theorem 3. *Assume the hypotheses*

(i) *The solution operator*

$$K(t, 0)x^0 = x^0 - \int_0^t A(s)x(s)\,ds - \int_0^t f(t, x)$$

is compact for $t > 0$.

(ii) $\sum_{j=1}^{\infty} a_{ij}(t)x_j + f_i(t, x) \geq 0$ *for every* $i = 1, 2, \cdots$, $x_j \geq 0$, $j = 1, 2, \cdots$.

Then system (6) *has a monotone nonincreasing solution* $x(t) = (x_1(t), x_2(t), \cdots)$ *such that* $x_i(t) \geq 0$ *for every* $i = 1, 2, \cdots$, $t \geq 0$, *and consequently the* $x_i(t)$ *are monotone nonincreasing.*

The proof follows as in Theorem 2, assuming that $\sum_{j=1}^{\infty} a_{ij}(t)x_j + f_i(t, x) > 0$ and the conclusion is that there exists a positive solution of (6) through some point $x_0 = (x_1^0, x_2^0, \cdots) \in \omega - S$. Consider then the system

$$\dot{y}_i + \Sigma(a_{ij}(t) + \varepsilon_{ij})x_j + f_i(t, x) = 0$$
$$y_i(0) = y_i^0$$

and a subsequence $y_{0,k_j}(t)$, $y_{0,k_j}(0) = y_{k_j}^0$ which converges, as $\varepsilon_{ij} \to 0$, uniformly on every interval $0 \leq t \leq T < \infty$. For system (6) the solution $x(t)$ can become zero after a finite time. Theorem 3 generalizes a result of Hartman and Wintner [3].

Example. Let $\{a_i\} \in c$ the space of convergent sequences with norm $\|a\| = \sup |a_i|$. Assume that $\lim_{i \to \infty} a_i = a_\infty \neq 0$ and define $\alpha^i = (0, 0, \cdots, a_i, 0, \cdots)$. Define $T(t)\alpha^i = \{e^{\lambda_i}\alpha^i\}$, $-\infty < \operatorname{Re}\lambda_i \leq \omega < \infty$. $T(t)$ is a strongly continuous semigroup with infinitesimal generator A given by $A\alpha^i = \{\lambda_i \alpha^i\}$. $T(t)$ is compact if and only if $\underline{\lim} \operatorname{Re}\lambda_i = -\infty$. Consider the system (8) $\dot{x}_i = -\lambda_i x_i - \sum_{j=1}^{\infty} g_{ij}(t)x_j$, $x_i(t_0) = x_i^0$ with $\lambda_i > 0$, $\sum_{i,j} \|g_{ij}\| < \infty$. This system can be written in the form $\dot{x} = Ax + G(t)x$, $x(t_0) = x^0$. Since $T(t)$ is compact and $G(t)$ is bounded, the consequent operator $K(t, t_0)x^0 = x^0 + \int_{t_0}^t (A + G(s))\,ds$ is compact. From Theorem 2 there exists at least one monotone solution $x(t) = (x_1(t), x_2(t), \cdots)$, of (8) such that $\lim_{t \to \infty} x(t) = 0$, $x(t) \geq 0$ and $-\dot{x}(t) \geq 0$.

References

1. K. Deimling, *Ordinary differential equations in Banach spaces*, Lecture Notes in Math., vol. 596, Springer-Verlag, 1977.
2. P. Hartman and A. Wintner, *Linear differential and difference equation with monotone solutions*, Amer. J. Math. **75** (1953), 731–743.
3. ———, *On monotone solutions of systems of nonlinear differential equations*, Amer. J. Math. (1954), 860–866.
4. D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math., vol. 840, Springer-Verlag, Berlin and New York.

5. A. F. Izé, *On a fixed point index method for the analysis of the asymptotic behavior and boundary value problems of infinite dimensional dynamical systems and processes.* J. Differential Equations **52** (1984).

6. J. Lindenstrauss and Z. Tzafriri, *Classical Banach space* I, *Sequence spaces,* Engebnisse der Mathematik und ihrer Grenzgebiete **92**, Springer-Verlag, Berlin, 1977.

7. T. Kato, *Quasilinear equations of evolution, with applications to partial differential equations,* Lecture Notes in Math., vol. 448, Springer-Verlag, 1974.

8. M. A. Krasnoselskii, *Positive solutions of operator equations.* Noordhoff, Groningen, 1964.

UFSCAR, Departamento de Matemática, Caixa Postal 676 , 13560 São Carlos, SP, Brasil