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ON THE VOLUME OF THE INTERSECTION OF TWO L_p^n BALLS

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ABSTRACT. In this note we prove that the proportion of the volume left in the L_p^n ball after removing a t-multiple of the L_q^n ball is of order $e^{-ct^p n^{p/q}}$.

1. INTRODUCTION

This note deals with the following problem, the case $p = 1$, $q = 2$ of which was raised by Vitali Milman: What is the volume left in the L_p^n ball after removing a t-multiple of the L_q^n ball? Recall that the L_r^n ball is the set $\{(t_1, t_2, \dots, t_n); t_i \in \mathbf{R}, n^{-1}\sum_{i=1}^n |t_i|^r \le 1\}$ and note that for $0 < p < q < \infty$ the L_q^n ball is contained in the L_p^n ball.

In Corollary 4 below, we show that after normalizing Lebesgue measure so that the volume of the L_p^n ball is one, the answer to the problem above is of order $e^{-ct^p n^{p/q}}$ for $T < t < \frac{1}{2}n^{1/p-1/q}$, where c and T depend on p and q but not on n.

The main theorem, Theorem 3, deals with the corresponding question for a certain measure on the L_p^n sphere. Theorem 3 and Corollary 4, together with some other remarks, form §3. In §2 we introduce a class of random variables to be used in proving the main theorem. These random variables are related to L_p in the same way that Gaussian variables are related to L_2.

2. PRELIMINARIES

Here we introduce a class of random variables to be used in the proof of the main theorem, and we summarize some of their properties. Fix a $0 < p < \infty$ and let $x, x_1, x_2, \dots, x_n$ be independent random variables each with density function $c_p e^{-t^p}$, $t > 0$. Note that necessarily $c_p = p/\Gamma(1/p)$. Let Δ_p denote

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the positive quadrant of the sphere of l_p^n, i.e.,

$$\Delta_p = \left\{(t_1, t_2, \dots, t_n); t_i \ge 0, \|t\|_p^p =: \sum t_i^p = 1\right\},$$

and for, λ, Lebesgue measure on $\mathbf{R}^n$, let $\overline{\mu}$ be the Borel probability on Δ_p given by

$$\overline{\mu}(A) = \frac{\lambda(\{t \in \mathbf{R}_+^n : \frac{t}{\|t\|_p} \in A, \quad \|t\|_p \le 1\})}{\lambda(\{t \in \mathbf{R}_+^n : \|t\|_p \le 1\})}.$$

Notice that for all $\varepsilon > 0$,

$$\overline{\mu}(A) = \frac{\lambda(\{(t_1, \dots, t_n) \in \mathbf{R}_+ \cdot A : a - \varepsilon < \|t\|_p \le a + \varepsilon\})}{\lambda(\{(t_1, \dots, t_n) \in \mathbf{R}_+^n : a - \varepsilon < \|t\|_p \le a + \varepsilon\})}.$$

Note that for $p = 1$ and $p = 2$ $\overline{\mu}$ is the normalized surface measure on Δ_p, and that for all p,

$$\overline{\nu}(A) = n \int_0^1 r^{n-1} \overline{\mu}\left(\frac{A}{r}\right) dr$$

where $\overline{\nu}$ denotes the normalized Lebesgue measure on the positive quadrant of the l_p^n ball. The first claim is known, though we could not locate a reference.

Lemma 1. *Put* $S = (\sum_{i=1}^n x_i^p)^{1/p}$ *; then* $(\frac{x_1}{S}, \frac{x_2}{S}, \dots, \frac{x_n}{S})$ *induces the measure* $\overline{\mu}$ *over the positive quadrant of the sphere of* l_p^n. *Moreover,* $(\frac{x_1}{S}, \frac{x_2}{S}, \dots, \frac{x_n}{S})$ *is independent of* S.

Proof. For any Borel subset A of Δ_p,

$$\begin{aligned}
&P\left(\left(\frac{x_1}{S}, \frac{x_2}{S}, \dots, \frac{x_n}{S}\right) \in A | S = a\right) \\
&= \lim_{\varepsilon \to 0} \frac{P((x_1, \dots, x_n) \in \mathbf{R}_+ A \ \& \ a - \varepsilon \le S \le a + \varepsilon)}{P(a - \varepsilon \le S \le a + \varepsilon)} \\
&= \lim_{\varepsilon \to 0} \int_{\substack{(t_1, \dots, t_n) \in \mathbf{R}_+ A \\ (a-\varepsilon)^p < \sum t_i^p < (a+\varepsilon)^p}} e^{-\sum t_i^p} dt \Big/ \int_{\substack{(t_1, \dots, t_n) \in \mathbf{R}_+^n \\ (a-\varepsilon)^p < \sum t_i^p < (a+\varepsilon)^p}} e^{-\sum t_i^p} dt \\
&\le \limsup_{\varepsilon \to 0} e^{-(a-\varepsilon)^p + (a-\varepsilon)^p} \int_{\substack{(t_1, \dots, t_n) \in \mathbf{R}_+ A \\ (a-\varepsilon)^p < \sum t_i^p < (a+\varepsilon)^p}} dt \Big/ \int_{\substack{(t_1, \dots, t_n) \in \mathbf{R}_+^n \\ (a-\varepsilon)^p < \sum t_i^p < (a+\varepsilon)^p}} dt \\
&= \overline{\mu}(A).
\end{aligned}$$

Similarly,

$$P\left(\left(\frac{x_1}{S}, \frac{x_2}{S}, \dots, \frac{x_n}{S}\right) \in A | S = a\right) \ge \overline{\mu}(A).$$

This proves that $P((\frac{x_1}{S}, \frac{x_2}{S}, \dots, \frac{x_n}{S}) \in A) = \overline{\mu}(A)$ and that $(\frac{x_1}{S}, \frac{x_2}{S}, \dots, \frac{x_n}{S})$ is independent of S. □

In the next claim we gather some more properties of the random variables x_i.

Lemma 2. *Let $x, x_1, \dots, x_n$ be as above; then*

1. *c_p is bounded away from zero and infinity when $p \to \infty$.*
2. *For all $h > 0$ and all $0 < p < \infty$, $\mathbf{E}e^{-hx^p} = (\frac{1}{1+h})^{1/p}$. In particular, $\mathbf{E}e^{-hx^p} \geq e^{-h/p}$ for all $h > 0$ and $\mathbf{E}e^{-hx^p} \leq e^{-h/2p}$ for all $0 < h \leq 1$.*
3. *For all $0 < u < \infty$ and all $0 < p < \infty$, $P(x^p > u) \geq \frac{c_p}{2p}e^{-2u}$. If $p \geq 1$ and $u \geq 1$, then also $P(x^p > u) \leq \frac{c_p}{p}e^{-u/2}$. In particular, for $p \geq 1$ and all u, $P(x^p > u) \leq Ce^{-u/2}$ for some universal C.*
4. *For all $1 \leq p \leq q < \infty$, if n is large enough, then $\mathbf{E}(\sum_{i=1}^n x_i^q)^{1/q}$ is equivalent, with universal constants, to $q^{1/p}n^{1/p}$, if $q \leq \log n$, and to $(\log n)^{1/p}$, otherwise.*

Proof.

(1) follows easily from the fact that $c_p = p/\Gamma(1/p) = \Gamma(\frac{1}{p}+1)^{-1}$.

(2) is a simple computation.

(3) is also simple; here is a sketch of the proof.

$$\begin{aligned} P(x^p > u) &= c_p \int_{u^{1/p}}^{\infty} e^{-t^p}\,dt \\ &\geq c_p \int_{u^{1/p}}^{(u+1)^{1/p}} \frac{pt^{(p-1)}}{p(u+1)^{(p-1)/p}} e^{-t^p}\,dt \\ &= \frac{c_p}{p(u+1)^{(p-1)/p}}\left(1-\frac{1}{e}\right)e^{-u} \\ &\geq \frac{c_p}{2p(u+1)}e^{-u} \\ &\geq \frac{c_p}{2p}e^{-2u}. \end{aligned}$$

The other inequality in (3) is proved in a similar way.

(4) First note that for all $0 < p$, $q < \infty$

$$\mathbf{E}x^q = c_p \int_0^\infty t^q e^{-t^p}\,dt = \frac{c_p}{p}\Gamma\left(\frac{q+1}{p}\right);$$

so that, by Holder's inequality and 1 above, if $1 \leq p \leq q < \infty$,

$$\mathbf{E}\left(\sum_{i=1}^n x_i^q\right)^{1/q} \leq \left(\sum_{i=1}^n \mathbf{E}x_i^q\right)^{1/q} = \left(\frac{c_p}{p}\Gamma\left(\frac{q+1}{p}\right)\right)^{1/q} n^{1/q} \leq Cq^{1/p}n^{1/q}$$

for some universal C. For the lower bound, note first that, by (3) above,

$$P\left(\max_{1\leq i\leq n} x_i > t\right) \geq 1 - \left(1 - \frac{c_p}{2p}e^{-2t^p}\right)^n.$$

For $n \geq 20p/c_p$, choose $t = 2^{1/p}(\log\frac{nc_p}{2p})^{1/p}$ to get that, for some universal c,

$$P\left(\max_{1\leq i\leq n} x_i > c(\log n)^{1/p}\right) \geq 1/2.$$

In particular, $\mathbf{E}\max_{1\le i\le n} x_i \ge c(\log n)^{1/p}$. This already implies the claim if $q > \log n$, since in this case $(\sum_{i=1}^n x_i^q)^{1/q}$ is universally equivalent to $\max_{1\le i\le n} x_i$. If $q \le \log n$, divide $\{1, 2, \dots, n\}$ into approximately n/e^q disjoint sets of cardinality approximately e^q each; then

$$\begin{aligned}\mathbf{E}\left(\sum_{i=1}^n x_i^q\right)^{1/q} &= \mathbf{E}\left(\sum_j \left(\sum_{i\in\sigma_j} x_i^q\right)^{q/q}\right)^{1/q} \\ &\ge \mathbf{E}\left(\sum_j \left(\max_{i\in\sigma_j} x_i\right)^q\right)^{1/q} \\ &\ge \left(\sum_j \left(\mathbf{E}\max_{i\in\sigma_j} x_i\right)^q\right)^{1/q} \\ &\ge c'(\log e^q)^{1/p}(n/e^q)^{1/q} \\ &\ge c'' q^{1/p} n^{1/q}. \quad \square\end{aligned}$$

The statement in Lemma 2.4, for the case $p = 2$, was noticed by the first-named author several years ago while seeking a precise estimate for the dimension of the Euclidean sections of l_p^n spaces (see [MS] p. 145, Remark 4.5). The original proof was more complicated. The proof presented here is an adaptation of a proof of the case $p = 2$ shown to us by J. Bourgain.

3. The main result

Theorem 3. *For all $1 \le p < q < \infty$ there are constants $c = c(p, q)$, $C = C(p, q)$, and $T = T(p, q)$ such that if μ denotes the probability measure on the positive quadrant of the unit sphere of L_p^n given by $\mu(A) = \overline{\mu}(n^{1/p}A)$, then*

$$\mu(\|u\|_{L_q^n} > t) \le \exp(-ct^p n^{p/q}) \tag{1}$$

for all $t > T$, and

$$\mu(\|u\|_{L_q^n} > t) \ge \exp(-Ct^p n^{p/q}) \tag{2}$$

for all $2 \le t \le \frac{1}{2}n^{1/p-1/q}$.

Moreover, for $q > 2p$ (or any other universal positive multiple of p), one can take $c(p, q) = \frac{\gamma}{p}$, $C(p, q) = \frac{\Gamma}{p}$, and $T(p, q) = \tau\min\{q, \log n\}^{1/p} \le q^{1/p}$. Here γ, Γ, and τ are universal constants.

Proof. By Lemma 1 above,

$$\mu(\|u\|_{L_q^n} > t) = P\left(n^{1/p-1/q}\left(\sum_{i=1}^n x_i^q\right)^{1/q} \Big/ \left(\sum_{i=1}^n x_i^p\right)^{1/p} > t\right),$$

where x_i are independent random variables each with density $c_p e^{-t^p}$. Assume, for the simplicity of the presentation, that n is even. Put $S = (\sum_{i=1}^n x_i^p)^{1/p}$ and let p_j, $j = 1, 2, \ldots, n/2$, be positive numbers with sum $\le 1/2$. Then

$$\begin{aligned} &P\left(n^{1/p-1/q}\left(\sum_{i=1}^n x_i^q\right)^{1/q} \Big/ \left(\sum_{i=1}^n x_i^p\right)^{1/p} > t\right) \\ (3)\qquad &= P\left(\sum_{i=1}^n x_i^q > \frac{t^q(\sum_{i=1}^n x_i^p)^{q/p}}{n^{q/p-1}}\right) \\ &\le \sum_{i=1}^{n/2} P(x_i^* > t p_i^{1/q} S/n^{1/p-1/q}) + P\left(\sum_{i=n/2+1}^{n} x_i^{*q} > t^q S^q/2n^{q/p-1}\right), \end{aligned}$$

where $\{x_j^*\}$ denotes the nonincreasing rearrangement of $\{|x_j|\}$.

Since

$$\begin{aligned} \sum_{j=n/2+1}^{n} x_i^{*q} \le \frac{n}{2} x_{n/2}^{*q} \le \frac{n}{2}\left(\frac{2}{n}\sum_{i=1}^{n/2} x_i^{*p}\right)^{q/p} \\ \le 2^{q/p-1} S^q/n^{q/p-1}, \end{aligned}$$

we get that, if $t \ge 2^{1/p}$, the second term in (3) is zero.

To evaluate the first term in (3), fix $1 \le j \le n/2$. Then,

$$\begin{aligned} P(x_j^* > t p_j^{1/q} S/n^{1/p-1/q}) &\le \binom{n}{j} P(x_1, \ldots, x_j > t p_j^{1/q} S/n^{1/p-1/q}) \\ &\le \binom{n}{j} P\left(x_1^p, \ldots, x_j^p > t^p p_j^{p/q} \sum_{i=j+1}^n x_i^p/n^{1-p/q}\right). \end{aligned}$$

From Lemma 2.3 and 2.2 we get that the last expression is dominated by

$$\begin{aligned} &\binom{n}{j} C^j \mathbf{E}\exp\left(-j p_j^{p/q} t^p \sum_{i=j+1}^n x_i^p/n^{1-p/q}\right) \\ &\le \binom{n}{j} C^j \exp(-j p_j^{p/q} t^p (n-j)/2pn^{1-p/q}) \end{aligned}$$

for some universal C. Note that the last inequality holds if $j n^{p/q-1} p_j^{p/q} t^p \le 1$. If this is not the case, the probability we are trying to evaluate is zero. Finally, the last term is dominated by

$$\exp\left(j\left(\log\frac{en}{j} + C - \frac{p_j^{p/q} t^p n^{p/q}}{4p}\right)\right). \tag{4}$$

Now, for α to be chosen momentarily, let p_j, $j = 1, \ldots, n/2$, be such that

$$j\left(\log\frac{en}{j} + C - \frac{p_j^{p/q} t^p n^{p/q}}{4p}\right) = -\alpha n^{p/q} t^p,$$

that is,

$$p_j = \left(4p\frac{\log\frac{en}{j}}{t^p n^{p/q}} + \frac{4Cp}{t^p n^{p/q}} + \alpha\frac{4p}{j}\right)^{q/p} .$$

We thus get that, for some universal constant C,

$$p_j \le 2^{q/p-1}\frac{(Cp)^{q/p}(\log\frac{en}{j})^{q/p}}{t^q n} + 2^{q/p-1}\alpha^{q/p}\frac{(4p)^{q/p}}{j^{q/p}} . \tag{5}$$

It is easy to see that, for $1 \le p < q < \infty$,

$$\sum_{j=1}^{n/2}\left(\log\frac{en}{j}\right)^{q/p} \le An\min\{nq^{q/p}, (\log n)^{q/p}\}$$

for some universal A. Thus the sum over the first terms in (5) is smaller than $1/4$ if, for some universal γ, $t > \gamma\min\{q^{1/p}, (\log n)^{1/p}\}$. The second term is bounded by $1/4$ if $\alpha < B\frac{1}{p}(\frac{q}{p}-1)^{p/q}$, for some universal B. Choosing α to satisfy this inequality and using (3), (4), and (5) we get that, for $t > \gamma\min\{q^{1/p}, (\log n)^{1/p}\}$,

$$\mu(\|u\|_{L_q^n} > t) \le \frac{n}{2}e^{-\alpha n^{p/q}t^p} .$$

Under the conditions on t, the factor $n/2$ can be absorbed in the second term (changing α to another constant of the same order of magnitude as a function of p), thus proving (1).

We now turn to the proof of the lower bound (2), which is simpler. Using Lemma 1 again,

$$\begin{aligned}
\mu(\|u\|_{L_q^n} > t) &= P\left(n^{1/p-1/q}\left(\sum_{i=1}^n x_i^q\right)^{1/q} \Big/ \left(\sum_{i=1}^n x_i^p\right)^{1/p} > t\right) \\
&\ge P(x_1 > St/n^{1/p-1/q}) \\
&= P\left(x_1 > \frac{t}{(n^{(1-p/q)} - t^p)^{1/p}}\left(\sum_{i=2}^n x_i^p\right)^{1/p}\right) .
\end{aligned}$$

Since $t^p \le \frac{1}{2}n^{(1-p/q)}$, this dominates

$$P\left(x_1 > \frac{2^{1/p}t}{n^{1/p-1/q}}\left(\sum_{i=2}^n x_i^p\right)^{1/p}\right) .$$

Now, by Lemma 2.3,

$$
\begin{aligned}
P\left(x_1 > \frac{2^{1/p}t}{n^{1/p-1/q}}\left(\sum_{i=2}^n x_i^p\right)^{1/p}\right) &\geq \frac{c_p}{2p}\mathbf{E}\exp\left(-4t^p\sum_{i=2}^n x_i^p/n^{(1-p/q)}\right) \\
&= \frac{c_p}{2p}(\mathbf{E}\exp(-4t^p x_1^p/n^{(1-p/q)}))^{n-1} \\
&= \frac{c_p}{2p}\left(\frac{1}{1+\frac{4t^p}{n^{1-p/q}}}\right)^{(n-1)/p} \qquad \text{(by Lemma 2.2)} \\
&\geq \frac{c_p}{2p}\exp\left(-\frac{4t^p(n-1)}{pn^{(1-p/q)}}\right) \\
&\geq \frac{c_p}{2p}e^{4t^p n^{p/q}/p}.
\end{aligned}
$$

Finally observe that, since c_p is bounded away from zero and $t \geq 2$, the factor $\frac{c_p}{2p}$ can be absorbed in the second term (changing 4 to another universal constant). □

Remarks. 1. It follows from the proof that, for n large enough and q close to p, one can take $c(p, q) = \frac{c}{p}(\frac{q}{p} - 1)$ for some universal constant c.

2. It follows from the statement of the theorem that, for $q = \infty$,

$$\mu(\|u\|_\infty > t) \leq e^{-\gamma t^p/p}$$

for all $t > \tau(\log n)^{1/p}$, and

$$\mu(\|u\|_\infty > t) \geq e^{-\Gamma t^p/p}$$

for all $2 \leq t \leq \frac{1}{2}n^{1/p}$, where γ, Γ, and τ are universal constants.

3. Note that it follows from Lemma 1 and Lemma 2.4 that the order of magnitude of T is the correct one.

4. The restriction $p \geq 1$ in Theorem 3 above and in Corollary 4 below can be replaced by $p > 0$ if one replaces the inequality $t \geq 2$ with $t \geq d$, for some d depending only on p and q, and if one removes the "moreover" part. We didn't check the dependence of the constants on p and q in this case.

5. The measure μ and the normalized Lebesgue measure are equivalent with constant at most $n^{1/2}$. It follows that a similar statement holds also for the normalized Lebesgue measure on the positive quadrant of the unit sphere of L_p^n. For $p = 1$ and $p = 2$ the measures $\overline{\mu}$ and normalized Lebesgue measure on the positive quadrant of the L_p^n sphere are equal.

Our last remark is that one can get a similar statement for the full balls. We state it as a corollary.

Corollary 4. *For all* $1 \le p < q < \infty$, *there are constants* $c = c(p, q)$, $C = C(p, q)$, *and* $T = T(p, q)$ *such that if* ν *denotes the normalized Lebesgue measure on the ball of* L_p^n, *then for all* n *large enough,*

$$\nu(\|u\|_{L_q^n} > t) \le \exp(-ct^p n^{p/q}) \tag{6}$$

for all $t > T$, *and*

$$\nu(\|u\|_{L_q^n} > t) \ge \exp(-Ct^p n^{p/q}) \tag{7}$$

for all $2 \le t \le \frac{1}{2} n^{1/p-1/q}$. *Moreover, for* $q > 2p$ *(or any other universal positive multiple of* p*), one can take* $c(p, q) = \frac{\gamma}{p}$, $C(p, q) = \frac{\Gamma}{p}$, *and* $T(p, q) = \tau \min\{q, \log n\}^{1/p} \le q^{1/p}$, *where* γ, Γ, *and* τ *are universal constants.*

The proof follows easily from Theorem 3 and the formula

$$\nu(A) = n \int_0^1 r^{n-1} \mu\left(\frac{A}{r}\right) dr,$$

which holds for all Borel sets A in the ball of L_p^n.

References

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