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THE STABILITY OF THE SINE AND COSINE FUNCTIONAL EQUATIONS

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ABSTRACT. In this work the stability of the functional equations describing the addition theorems for sine and cosine is proved.

1. INTRODUCTION

The aim of this paper is to study the stability properties of two well known functional equations: the sine and cosine equations. Stability problems concerning classical functional equations have been treated by several authors (see e.g. [1, 2, 3, 4, 5]). Here we use some results of [3, 4].

If G is a semigroup and $\mathbf{K}$ is a field, then functions a, $m: G \to \mathbf{K}$ satisfying the functional equation

$$a(xy) = a(x) + a(y)$$

resp.

$$m(xy) = m(x)m(y)$$

will be called additive, resp. exponential.

In this work $\mathbf{K}$ denotes either the real or the complex field. We remark that all the lemmata are valid for arbitrary field $\mathbf{K}$.

If G is a semigroup, $\mathbf{K}$ is a field, and $\mathscr{F}$ is a linear space of $\mathbf{K}$-valued functions on G, then we say that the functions f, $g: G \to \mathbf{K}$ are linearly independent modulo $\mathscr{F}$ if $\lambda f + \mu g \in \mathscr{F}$ implies $\lambda = \mu = 0$ for any λ, μ in $\mathbf{K}$. We say that the linear space $\mathscr{F}$ is two-sided invariant if $f \in \mathscr{F}$ implies that the functions $x \to f(xy)$ and $x \to f(yx)$ belong to $\mathscr{F}$ for any y in G.

2. STABILITY OF THE SINE EQUATION

Lemma 2.1. *Let G be a semigroup, f, $g: G \to \mathbf{K}$ be functions, and $\mathscr{F}$ be a two-sided invariant linear space of $\mathbf{K}$-valued functions on G. Suppose that f and g are linearly independent modulo $\mathscr{F}$. If the function*

$$x \to f(xy) - f(x)g(y) - f(y)g(x)$$

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belongs to $\mathscr{F}$ *for all* y *in* G, *then*

$$f(xy) = f(x)g(y) + f(y)g(x)$$

holds for all x, y *in* G.

Proof. We define

$$F(x, y) = f(xy) - f(x)g(y) - f(y)g(x)$$

for x, y in G; then there are constants $\lambda_0, \lambda_1, \lambda_2$ and $y_1 \in G$ with

$$g(x) = \lambda_0 f(x) + \lambda_1 f(xy_1) + \lambda_2 F(x, y_1)$$

for all x in G. Now we have for all x, y, z in G

$$\begin{aligned} f[(xy)z] &= f(xy)g(z) + f(z)g(xy) + F(xy, z) \\ &= f(x)g(y)g(z) + g(x)f(y)g(z) \\ &\quad + F(x, y)g(z) + \lambda_0 f(xy)f(z) + \lambda_1 f(xyy_1)f(z) \\ &\quad + \lambda_2 F(xy, y_1)f(z) + F(xy, z) \\ &= f(x)g(y)g(z) + g(x)f(y)g(z) + F(x, y)g(z) \\ &\quad + \lambda_0 f(x)g(y)f(z) + \lambda_0 g(x)f(y)f(z) + \lambda_0 F(x, y)f(z) \\ &\quad + \lambda_1 f(x)g(yy_1)f(z) + \lambda_1 g(x)f(yy_1)f(z) + \lambda_1 F(x, yy_1)f(z) \\ &\quad + \lambda_2 F(xy, y_1)f(z) + F(xy, z). \end{aligned}$$

On the other hand,

$$f[(xy)z] = f[x(yz)] = f(x)g(yz) + g(x)f(yz) + F(x, yz).$$

It follows that

$$\begin{aligned} &f(x)[g(y)g(z) + \lambda_0 g(y)f(z) + \lambda_1 g(yy_1)f(z) - g(yz)] \\ &\qquad + g(x)[f(y)g(z) + \lambda_0 f(y)f(z) + \lambda_1 f(yy_1)f(z) - f(yz)] \\ &\quad = F(x, yz) - F(xy, z) - F(x, y)g(z) - \lambda_0 F(x, y)f(z) \\ &\qquad - \lambda_1 F(x, yy_1)f(z) - \lambda_2 F(xy, y_1)f(z). \end{aligned}$$

Using the linear independence of f and g modulo $\mathscr{F}$ and also the fact that $\mathscr{F}$ is a two-sided invariant linear space, we have

$$\begin{aligned} g(y)g(z) + [\lambda_0 g(y) + \lambda_1 g(yy_1)]f(z) &= g(yz) \\ f(y)g(z) + [\lambda_0 f(y) + \lambda_1 f(yy_1)]f(z) &= f(yz) \end{aligned}$$

for all y, z in G. The second equation can be rewritten as

$$f(y)g(z) + [g(y) - \lambda_2 F(y, y_1)]f(z) = f(yz)$$

and, on the other hand, we have

$$f(y)g(z) + g(y)f(z) + F(y, z) = f(yz).$$

This implies

$$F(y, z) = -\lambda_2 F(y, y_1)f(z)$$

for all y, z in G. Further, with notation

$$\psi(x) = \lambda_0 g(x) + \lambda_1 g(xy_1)$$

we have the following

$$\begin{aligned} f[(xy)z] &= f(xy)g(z) + f(z)[g(xy) - \lambda_2 F(xy, y_1)] \\ &= f(x)g(y)g(z) + f(y)[g(x) - \lambda_2 F(x, y_1)]g(z) \\ &\quad + g(x)g(y)f(z) + \psi(x)f(y)f(z) - \lambda_2 F(xy, y_1)f(z) \\ &= f(x)g(y)g(z) + g(x)f(y)g(z) + g(x)g(y)f(z) \\ &\quad + \psi(x)f(y)f(z) - \lambda_2 F(x, y_1)f(y)g(z) - \lambda_2 F(xy, y_1)f(z). \end{aligned}$$

On the other hand

$$f[(xy)z] = f[x(yz)] = f(x)g(yz) + g(x)f(yz) + F(x, yz),$$

and hence the above chain of equalities can be continued as

$$\begin{aligned} f[(xy)z] &= f[x(yz)] = f(x)g(yz) + [g(x) - \lambda_2 F(x, y_1)]f(yz) \\ &= f(x)g(y)g(z) + f(x)\psi(y)f(z) + g(x)f(y)g(z) + g(x) \\ &\quad \times [g(y) - \lambda_2 F(y, y_1)]f(z) - \lambda_2 F(x, y_1)f(y)g(z) \\ &\quad - \lambda_2 F(x, y_1([g(y) - \lambda_2 F(y, y_1)]f(z) \\ &= f(x)g(y)g(z) + g(x)f(y)g(z) + g(x)g(y)f(z) + f(x)\psi(y)f(z) \\ &\quad - \lambda_2 F(y, y_1)g(x)f(z) - \lambda_2 F(x, y_1)f(y)g(z) \\ &\quad - \lambda_2 F(x, y_1)g(y)f(z) + \lambda_2^2 F(x, y_1)F(y, y_1)f(z). \end{aligned}$$

As $f \neq 0$, we can divide by $f(z)$ to obtain

$$\begin{aligned} &\psi(x)f(y) - \lambda_2 F(xy, y_1) \\ &\quad = f(x)\psi(y) - \lambda_2 F(y, y_1)g(x) - \lambda_2 F(x, y_1)g(y) + \lambda_2^2 F(x, y_1)F(y, y_1). \end{aligned}$$

Interchanging x and y, we have

$$\begin{aligned} &\psi(y)f(x) - \lambda_2 F(yx, y_1) \\ &\quad = f(y)\psi(x) - \lambda_2 F(x, y_1)g(y) - \lambda_2 F(y, y_1)g(x) + \lambda_2^2 F(y, y_1)F(x, y_1). \end{aligned}$$

By adding the two equations, we obtain

$$\begin{aligned} &-\lambda_2[F(xy, y_1) + F(yx, y_1)] \\ &\quad = -2\lambda_2 F(y, y_1)g(x) - 2\lambda_2 F(x, y_1)g(y) + 2\lambda_2^2 F(x, y_1)F(y, y_1). \end{aligned}$$

As g is not in $\mathcal{F}$, it follows that $\lambda_2 F(y, y_1) = 0$; hence, $F = 0$.

Lemma 2.2. *Let G be a semigroup, $f, g\colon G \to \mathbf{K}$ functions and let $\mathcal{F}$ be a two-sided invariant linear space of* $\mathbf{K}$*-valued functions on G. If the function*

$$x \to f(xy) - f(x)g(y) - f(y)g(x)$$

belongs to $\mathcal{F}$ for all y in G, then we have the following possibilities:

(i) $f = 0$, *g is arbitrary*;

(ii) $f, g \in \mathscr{F}$;
(iii) $g \in \mathscr{F}$ *is an exponential*;
(iv) $f = \lambda m - \lambda b$, $g = \frac{1}{2}m + \frac{1}{2}b$, *where* $m: G \to \mathbf{K}$ *is an exponential,* $b: G \to \mathbf{K}$ *is in* $\mathscr{F}$ *and* $\lambda \in \mathbf{K}$ *is a constant*;
(v) $f(xy) = f(x)g(y) + f(y)g(x)$ *for all* x, y *in* G.

Proof. If f and g are linearly independent modulo $\mathscr{F}$, then (v) follows from Lemma 2.1.

Now we suppose that there are constants μ, ν in $\mathbf{K}$ (at least one of them different from zero) such that $\mu f + \nu g \in \mathscr{F}$ but $f, g \notin \mathscr{F}$. Then we have $g = \frac{1}{2\lambda} f + b$ with $b \in \mathscr{F}$ and $\lambda \neq 0$, hence the function

$$x \to f(xy) - \left[\frac{1}{\lambda} f(y) + b(y)\right] f(x)$$

belongs to $\mathscr{F}$ for all y in G. From the results of [3] it follows that

$$\frac{1}{\lambda} f(y) + b(y) = m(y)$$

where $m: G \to \mathbf{K}$ is an exponential, which implies (iv).

If $g \in \mathscr{F}$ and $f \notin \mathscr{F}$, then

$$x \to f(xy) - f(x)g(y)$$

belongs to $\mathscr{F}$ for all y in G, and it follows from [3] that (iii) holds.

If $f \in \mathscr{F}$ and $f \neq 0$, then $g \in \mathscr{F}$.

If $f = 0$, then g is arbitrary.

Theorem 2.3. *Let* G *be an amenable group and let* $f, g: G \to \mathbf{K}$ *be given functions. The function*

$$(x, y) \to f(xy) - f(x)g(y) - f(y)g(x)$$

is bounded if and only if we have one of the following conditions:

(i) $f = 0$, g *is arbitrary*;
(ii) f, g, *are bounded*;
(iii) $f = am + b$, $g = m$, *where* $a: G \to \mathbf{K}$ *is additive,* $m: G \to \mathbf{K}$ *is a bounded exponential and* $b: G \to \mathbf{K}$ *is a bounded function*;
(iv) $f = \lambda m - \lambda b$, $g = \frac{1}{2}m + \frac{1}{2}b$, *where* $m: G \to \mathbf{K}$ *is an exponential,* $b: G \to \mathbf{K}$ *is a bounded function, and* $\lambda \in \mathbf{K}$ *is a constant*;
(v) $f(xy) = f(x)g(y) + f(y)g(x)$, *for all* x, y *in* G.

Proof. Applying Lemma 2.2 with $\mathscr{F}$ denoting the set of all bounded $\mathbf{K}$-valued functions on G we see that either one of the above conditions (i), (ii), (iv), (v) is fulfilled, or $g = m$ is a bounded exponential. In the latter case the function

$$(x, y) \to f(xy)m((xy)^{-1}) - f(x)m(x^{-1}) - f(y)m(y^{-1})$$

is bounded, hence by Hyers's theorem [4]

$$f(x)m(x^{-1}) = a(x) + b_0(x)$$

holds for all x in G where $a: G \to \mathbf{K}$ is additive and $b_0: G \to \mathbf{K}$ is bounded, and our statement follows. (We have excluded the trivial case $m = 0$, and we have used the obvious identity $m(x)m(x^{-1}) = 1$, which holds for any nonzero exponential.)

The sufficiency follows by direct calculation.

3. Stability of the cosine equation

Lemma 3.1. *Let G be a semigroup, $f, g: G \to \mathbf{K}$ be functions, and $\mathscr{F}$ be a two-sided invariant linear space of $\mathbf{K}$-valued functions on G. Suppose that f and g are linearly independent modulo $\mathscr{F}$. If the functions*

$$x \to f(xy) - f(x)f(y) + g(x)g(y)$$

and

$$x \to f(xy) - f(yx)$$

belong to $\mathscr{F}$ for all y in G, then

$$f(xy) = f(x)f(y) - g(x)g(y)$$

holds for all x, y in G.

Proof. We define

$$F(x, y) = f(xy) - f(x)f(y) + g(x)g(y)$$

for x, y in G; then there are constants $\lambda_0, \lambda_1, \lambda_2$ and $y_1 \in G$ with

$$g(x) = \lambda_0 f(x) + \lambda_1 f(xy_1) + \lambda_2 F(x, y_1)$$

for all x in G. Now we have, for all x, y, z in G,

$$\begin{aligned}
f[(xy)z] &= f(xy)f(z) - g(z)g(xy) + F(xy, z) \\
&= f(x)f(y)f(z) - g(x)g(y)f(z) + F(x, y)f(z) \\
&\quad - \lambda_0 f(xy)g(z) - \lambda_1 f(xyy_1)g(z) - \lambda_2 F(xy, y_1)g(z) + F(xy, z) \\
&= f(x)f(y)f(z) - g(x)g(y)f(z) + F(x, y)f(z) \\
&\quad - \lambda_0 f(x)f(y)g(z) + \lambda_0 g(x)g(y)g(z) - \lambda_0 F(x, y)g(z) \\
&\quad - \lambda_1 f(x)f(yy_1)g(z) + \lambda_1 g(x)g(yy_1)g(z) - \lambda_1 F(x, yy_1)g(z) \\
&\quad - \lambda_2 F(xy, y_1)g(z) + F(xy, z).
\end{aligned}$$

On the other hand, as above,

$$f[(xy)z] = f[x(yz)] = f(x)f(yz) - g(x)g(yz) + F(x, yz),$$

and it follows that

$$\begin{aligned}
&f(x)[f(y)f(z) - \lambda_0 f(y)g(z) - \lambda_1 f(yy_1)g(z) - f(yz)] \\
&\qquad - g(x)[g(y)f(z) - \lambda_0 g(y)g(z) - \lambda_1 g(yy_1)g(z) - g(yz)] \\
&\quad = F(x, yz) - F(xy, z) - F(x, y)f(z) + \lambda_0 F(x, y)g(z) \\
&\qquad + \lambda_1 F(x, yy_1)g(z) + \lambda_2 F(xy, y_1)g(z).
\end{aligned}$$

Using the linear independence of f and g modulo $\mathscr{F}$, we have

$$F(x, yz) - F(xy, z) = F(x, y)f(z) - \lambda_0 F(x, y)g(z) \\ - \lambda_1 F(x, yy_1)g(z) - \lambda_2 F(xy, y_1)g(z).$$

By the assumptions of the lemma, it follows that the left-hand side belongs to $\mathscr{F}$ as a function of z for all fixed x, y in G. Again using the linear independence of f and g modulo $\mathscr{F}$ and the fact that $\mathscr{F}$ is a two-sided invariant linear space, we have $F(x, y) = 0$ for all x, y in G; hence, the lemma is proved.

Lemma 3.2. *Let G be a semigroup, $f, g\colon G \to \mathbf{K}$ functions and let $\mathscr{F}$ be a two-sided invariant linear space of $\mathbf{K}$-valued functions on G. If the functions*

$$x \to f(xy) - f(x)f(y) + g(x)g(y)$$

and

$$x \to f(xy) - f(yx)$$

belong to $\mathscr{F}$ for all y in G, then we have the following possibilities:

(i) $f, g \in \mathscr{F}$;

(ii) *f is an exponential, $g \in \mathscr{F}$;*

(iii) *$f + g$ or $f - g$ is an exponential in $\mathscr{F}$;*

(iv) *$f = \frac{\lambda^2}{\lambda^2-1}m - \frac{1}{\lambda^2-1}b$, $g = \frac{\lambda}{\lambda^2-1}m + \frac{\lambda}{\lambda^2-1}b$, where $m\colon G \to \mathbf{K}$ is an exponential, $b\colon G \to \mathbf{K}$ is in $\mathscr{F}$, and $\lambda \in \mathbf{K}$ is a constant with $\lambda^2 \neq 1$;*

(v) $f(xy) = f(x)f(y) - g(x)g(y)$.

Proof. If f and g are linearly independent modulo $\mathscr{F}$, then (v) follows from Lemma 3.1.

If $g \in \mathscr{F}$, then (i) or (ii) follows from [3].

If $f \in \mathscr{F}$, then $g \in \mathscr{F}$, hence (i) follows.

Now we suppose that f are g are linearly dependent modulo $\mathscr{F}$, but $f, g \notin \mathscr{F}$. Then there exists a constant $\lambda \neq 0$ with $f = \lambda g + b$ and $b \in \mathscr{F}$; hence, by our assumpticn, the function

$$x \to g(xy) - \tfrac{1}{\lambda}[(\lambda^2 - 1)g(y) + \lambda b(y)]g(x)$$

belongs to $\mathscr{F}$. By [3], we have that

$$\frac{\lambda^2 - 1}{\lambda}g + b = m$$

is an exponential, which gives (iii) for $\lambda^2 = 1$ and (iv) for $\lambda^2 \neq 1$. The iemma is proved.

Theorem 3.3. *Let G be an amenable group and let $f, g\colon G \to \mathbf{K}$ be given functions. The function*

$$(x, y) \to f(xy) - f(x)f(y) + g(x)g(y)$$

is bounded if and only if we have one of the following possibilities:

(i) *f, g, are bounded;*

(ii) *f is an exponential, g is bounded*;

(iii) $f = (1+a)m + b$, $g = am + b$ *or* $f = am + b$, $g = (1-a)m - b$, *where* $a: G \to \mathbf{K}$ *is additive,* $m: G \to \mathbf{K}$ *is a bounded exponential, and* $b: G \to \mathbf{K}$ *is a bounded function*;

(iv) $f = \frac{\lambda^2}{\lambda^2-1}m - \frac{1}{\lambda^2-1}b$, $g = \frac{\lambda}{\lambda^2-1}m + \frac{\lambda}{\lambda^2-1}b$, *where* $m: G \to \mathbf{K}$ *is an exponential,* $b: G \to \mathbf{K}$ *is a bounded function, and* $\lambda \in \mathbf{K}$ *is a constant with* $\lambda^2 \neq 1$;

(v) $f(xy) = f(x)f(y) - g(x)g(y)$ *for all* x, y *in* G.

Proof. Again we let $\mathscr{F}$ be the set of all bounded $\mathbf{K}$-valued functions on G, and we apply Lemma 3.2. First we prove the necessity. If g is bounded, then we have (i) or (ii) by [3]. If $f+g$ or $f-g$ is a bounded exponential, which corresponds to (iii) of Lemma 3.2, then we have (iii) by using Hyers's theorem as in Theorem 2.3. Finally, the rest follows directly by Lemma 3.2.

The sufficiency follows by direct calculation.

4. Remark

The above results show that both the sine and cosine equations have the remarkable stability property that the difference between the two sides of the equation remains bounded if and only if some bounded functions are added to the exact solutions.

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