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COMMUTATIVE BANACH ALGEBRAS WHICH SATISFY A BOCHNER-SCHOENBERG-EBERLEIN-TYPE THEOREM

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Dedicated to Professor Junzo Wada

ABSTRACT. A class of commutative Banach algebras which satisfy a Bochner-Schoenberg-Eberlein-type inequality is introduced. Commutative C^*-algebras, the disk algebra and the Hardy algebra on the open disk are examples.

1. INTRODUCTION

In this paper we study commutative Banach algebras satisfying a Bochner-Schoenberg-Eberlein-type theorem. Let G be a locally compact Abelian group. Let $G^\frown$ denote the dual group of G and $M(G)$ denote the measure algebra of G as usual. The algebra of all bounded continuous complex-valued functions on $G^\frown$ will be denoted by $C^b(G^\frown)$ and the Fourier-Stieltjes transform of μ in $M(G)$ by $\mu^\frown$. A theorem of Bochner-Schoenberg-Eberlein for the group algebra $L^1(G)$ states the following (cf. [10]):

Theorem (Bochner-Schoenberg-Eberlein). *Let σ be a function in $C^b(G^\frown)$ and β be a positive real number. The following are equivalent:*

(i) *There exists μ in $M(G)$ which satisfies $\sigma = \mu^\frown$ and $\|\mu\| \le \beta$.*

(ii) *For every finite number of complex numbers $c_1, \dots, c_n$ and the same number of $\gamma_1, \dots, \gamma_n$ in $G^\frown$ the inequality*

$$\left|\sum_{i=1}^{n} c_i\sigma(\gamma_i)\right| \le \beta \left\|\sum_{i=1}^{n} c_i\gamma_i\right\|_{L^\infty(G)}$$

holds.

Let A be a (not necessary unital) commutative Banach algebra without order (i.e. which has no nonzero annihilators). A^* denotes the dual space of A and Φ_A the carrier space of A. We denote by $C(\Phi_A)$ the algebra consisting of all continuous complex-valued functions on Φ_A and $C_{\mathrm{BSE}}(\Phi_A)$ by the set of all

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σ in $C(\Phi_A)$ which satisfy the following: there exists a positive real number β such that for every finite number of complex numbers $c_1, \dots, c_n$ and the same number of $\varphi_1, \dots, \varphi_n$ in Φ_A the inequality

$$\left|\sum_{i=1}^{n} c_i\sigma(\varphi_i)\right| \leq \beta \left\|\sum_{i=1}^{n} c_i\varphi_i\right\|_{A^*}$$

holds. Also for each σ in $C_{\mathrm{BSE}}(\Phi_A)$ we denote by $\|\sigma\|_{\mathrm{BSE}}$ the infimum of such β. Then we can easily see that for each $x \in A$, its Gelfand transform $x^\frown$ belongs to $C_{\mathrm{BSE}}(\Phi_A)$ and $\|x^\frown\|_\infty \leq \|x^\frown\|_{\mathrm{BSE}}$. Here $\| \ \|_\infty$ denotes the supremum norm on Φ_A. Also we have the following:

Lemma 1.

(i) *$C_{\mathrm{BSE}}(\Phi_A)$ is a subalgebra of $C(\Phi_A)$.*
(ii) *$\| \ \|_{\mathrm{BSE}}$ is a complete algebra norm on $C_{\mathrm{BSE}}(\Phi_A)$.*
(iii) *$C_{\mathrm{BSE}}(\Phi_A)$ is semisimple.*

Proof. (i) We can easily see that $C_{\mathrm{BSE}}(\Phi_A)$ is a linear subspace of $C(\Phi_A)$. Then we only show that the product of two functions in $C_{\mathrm{BSE}}(\Phi_A)$ is another one. To do this let σ_1, $\sigma_2 \in C_{\mathrm{BSE}}(\Phi_A)$, $c_1, \dots, c_n$ be complex numbers and $\varphi_1, \dots, \varphi_n \in \Phi_A$. For any $\varepsilon > 0$, choose $x_1 \in A$ such that $\|x_1\| \leq 1$ and

$$\left\|\sum_{i=1}^{n} c_i\sigma_1(\varphi_i)\varphi_i\right\|_{A^*} \leq \left|\sum_{i=1}^{n} c_i\sigma_1(\varphi_i)\varphi_i(x_1)\right| + \varepsilon.$$

Moreover choose $x_2 \in A$ such that $\|x_2\| \leq 1$ and

$$\left\|\sum_{i=1}^{n} c_i\varphi_i(x_1)\varphi_i\right\|_{A^*} \leq \left|\sum_{i=1}^{n} c_i\varphi_i(x_1)\varphi_i(x_2)\right| + \varepsilon.$$

Then we have

$$\begin{aligned}
&\left|\sum_{i=1}^{n} c_i(\sigma_1\sigma_2)(\varphi_i)\right| \\
&\quad \leq \|\sigma_2\|_{\mathrm{BSE}} \left\|\sum_{i=1}^{n} c_i\sigma_1(\varphi_i)\varphi_i\right\|_{A^*} \\
&\quad \leq \|\sigma_2\|_{\mathrm{BSE}} \left(\left|\sum_{i=1}^{n} c_i\varphi_i(x_1)\sigma_1(\varphi_i)\right| + \varepsilon\right) \\
&\quad \leq \|\sigma_2\|_{\mathrm{BSE}} \left(\|\sigma_1\|_{\mathrm{BSE}} \left\|\sum_{i=1}^{n} c_i\varphi_i(x_1)\varphi_i)\right\|_{A^*} + \varepsilon\right) \\
&\quad \leq \|\sigma_2\|_{\mathrm{BSE}} \left(\|\sigma_1\|_{\mathrm{BSE}} \left(\left|\sum_{i=1}^{n} c_i\varphi_i(x_1)\varphi_i(x_2)\right| + \varepsilon\right) + \varepsilon\right) \\
&\quad \leq \|\sigma_2\|_{\mathrm{BSE}} \left(\|\sigma_1\|_{\mathrm{BSE}} \left(\left\|\sum_{i=1}^{n} c_i\varphi_i\right\|_{A^*} \|x_1x_2\| + \varepsilon\right) + \varepsilon\right).
\end{aligned}$$

Since $\|x_1x_2\| \leq 1$ and ε is arbitrary, it follows that $\sigma_1\sigma_2 \in C_{\mathrm{BSE}}(\Phi_A)$ and $\|\sigma_1\sigma_2\|_{\mathrm{BSE}} \leq \|\sigma_1\|_{\mathrm{BSE}}\|\sigma_2\|_{\mathrm{BSE}}$.

(ii) Observe that $\| \ \|_{\mathrm{BSE}}$ is an algebra norm on $C_{\mathrm{BSE}}(\Phi_A)$ by the same argument of the above proof. Then we only show that $\| \ \|_{\mathrm{BSE}}$ is complete. Suppose that $\sigma_n \in C_{\mathrm{BSE}}(\Phi_A)$ $(n = 1, 2, \ldots)$ and $\lim_{n,m\to\infty} \|\sigma_n - \sigma_m\|_{\mathrm{BSE}} = 0$. Then for each $\varphi \in \Phi_A$, $|\sigma_n(\varphi) - \sigma_m(\varphi)| \leq \|\sigma_n - \sigma_m\|_{\mathrm{BSE}}$ $(n, m = 1, 2, \ldots)$ and hence $\{\sigma_n(\varphi)\}$ converges to a complex number, say $\sigma(\varphi)$. In this case, we can see that $\{\sigma_n(\varphi)\}$ converges uniformly to $\sigma(\varphi)$ on Φ_A. Therefore, σ is a continuous function on Φ_A. Now let $c_1, \ldots, c_k$ be complex numbers and $\varphi_1, \ldots, \varphi_k \in \Phi_A$. For any $\varepsilon > 0$, choose a number N such that $\|\sigma_n - \sigma_m\|_{\mathrm{BSE}} \leq \varepsilon$ for all n, $m \geq N$. Then we have

$$\left|\sum_{i=1}^{k} c_i(\sigma_n(\varphi_i) - \sigma_m(\varphi_i))\right| \leq \|\sigma_n - \sigma_m\|_{\mathrm{BSE}} \left\|\sum_{i=1}^{k} c_i\varphi_i\right\|_{A^*} < \varepsilon \left\|\sum_{i=1}^{k} c_i\varphi_i\right\|_{A^*}$$

for all n, $m \geq N$. Hence, by taking the limit with respect to m, we obtain that

$$\left|\sum_{i=1}^{k} c_i(\sigma_n(\varphi_i) - \sigma(\varphi_i))\right| \leq \varepsilon \left\|\sum_{i=1}^{k} c_i\varphi_i\right\|_{A^*}$$

for all $n \geq N$. This shows that $\sigma \in C_{\mathrm{BSE}}(\Phi_A)$ and $\lim_{n\to\infty} \|\sigma_n - \sigma\|_{\mathrm{BSE}} = 0$.

(iii) Observe that evaluations (at points of Φ_A) are nonzero multiplicative linear functionals and the only point in $C_{\mathrm{BSE}}(\Phi_A)$ annihilated by all of them is 0. Then $C_{\mathrm{BSE}}(\Phi_A)$ is semisimple. Q.E.D.

Remark. By the above argument, $C_{\mathrm{BSE}}(\Phi_A)$ is a semisimple commutative Banach algebra with respect to the BSE norm and contains $A^\frown = \{x^\frown : x \in A\}$.

A multiplier T on A is the bounded linear operator on A into itself which satisfies $xTy = (Tx)y$ for every x and y in A. $M(A)$ denotes the commutative Banach algebra consisting of all multipliers on A. It is well known that T can be represented as a continuous complex-valued function $T^\frown$ on Φ_A (cf. [7]). We denote $M^\frown(A) = \{T^\frown : T \in M(A)\}$. Along these lines, the preceding theorem of Bochner–Schoenberg–Eberlein for a locally compact Abelian group is rewritten as $M^\frown(L^1(G)) = C_{\mathrm{BSE}}(\Phi_{L^1(G)})$, since we know that $L^\infty(G) \cong L^1(G)^*$, $G^\frown \cong \Phi_{L^1(G)}$ and $M(G) \cong M(L^1(G))$. We will discuss commutative Banach algebras with such a condition. For this purpose we make the following

Definition. We say that a commutative Banach algebra A without order is a BSE-algebra if A satisfies the condition $M^\frown(A) = C_{\mathrm{BSE}}(\Phi_A)$.

Remark. Precisely speaking the theorem of Bochner–Schoenberg–Eberlein in terms of BSE, we see that $M^\frown(A) = C_{\mathrm{BSE}}(\Phi_A)$ and $\|T\| = \|T^\frown\|_{\mathrm{BSE}}$ for all $T \in M(A)$ whenever $A = L^1(G)$ (G: locally compact Abelian group). However we see the following fact from Corollary 6 later on: if A is a BSE-algebra, then A is semisimple if and only if there exist positive real numbers α and β such that $\alpha\|T\| \leq \|T^{\wedge}\|_{\text{BSE}} \leq \beta\|T\|$ for all $t \in M(A)$. Of course $L^1(G)$ is semisimple. Therefore it seems reasonable to define BSE-algebras as in the above in view of topological isomorphism.

2. BSE-ALGEBRAS

Let $C^b(\Phi_A)$ be the algebra of all bounded continuous complex-valued functions on Φ_A. Then $C_{\text{BSE}}(\Phi_A)$ is contained in $C^b(\Phi_A)$ and hence an arbitrary BSE-algebra can have the following two types:

$$\text{(I)} \quad M^{\wedge}(A) = C_{\text{BSE}}(\Phi_A) = C^b(\Phi_A).$$
$$\text{(II)} \quad M^{\wedge}(A) = C_{\text{BSE}}(\Phi_A) \subsetneqq C^b(\Phi_A).$$

For example, the group algebra of a finite group is type I-BSE and the group algebra of an infinite group is type II-BSE. In this section we show that a certain type I-BSE-algebra is isomorphic to a commutative C^*-algebra. We also show that both the disk algebra and the classical Hardy algebra are type II-BSE.

Now we begin with the following

Lemma 2. $C_{\text{BSE}}(\Phi_A) = C^b(\Phi_A)$ *if and only if there exists* $\beta < \infty$ *such that for any finite number of* $c_1, \dots, c_n \in \Delta$ *and the same number of* $\varphi_1, \dots, \varphi_n \in \Phi_A$, *there exists* x *in* A *such that* $\|x\| \leq \beta$ *and* $x^{\wedge}(\varphi_i) = c_i$ $(i = 1, \dots, n)$. *Here* Δ *denotes the closed unit disk.*

Proof. We first prove the "only if" part. Suppose that $C_{\text{BSE}}(\Phi_A) = C^b(\Phi_A)$. Since the inequality $\|\sigma\|_\infty \leq \|\sigma\|_{\text{BSE}}$ holds for every $\sigma \in C_{\text{BSE}}(\Phi_A)$, it follows from the open mapping theorem that there exists $\beta < \infty$ such that $\|\sigma\|_{\text{BSE}} \leq \beta\|\sigma\|_\infty$ for every $\sigma \in C^b(\Phi_A)$. Consider a finite number of $c_1, \dots, c_n$ in Δ and the same number of $\varphi_1, \dots, \varphi_n$ in Φ_A. Choose a function σ in the unit ball of $C^b(\Phi_A)$ with $\sigma(\varphi_i) = c_i$ for every $i = 1, \dots, n$. By Helly's theorem (cf. [8]) there exists $x \in A$ such that $\|x\| \leq \beta + 1$ and $x^{\wedge}(\varphi_i) = c_i$ for $i = 1, \dots, n$, since for any complex numbers $a_1, \dots, a_n$, the inequalities

$$\begin{aligned} \left|\sum_{i=1}^n a_i c_i\right| &= \left|\sum_{i=1}^n a_i \sigma(\varphi_i)\right| \leq \|\sigma\|_{\text{BSE}} \left\|\sum_{i=1}^n a_i \varphi_i\right\|_{A^*} \\ &= \beta\|\sigma\|_\infty \left\|\sum_{i=1}^n a_i \varphi_i\right\|_{A^*} \\ &= \beta \left\|\sum_{i=1}^n a_i \varphi_i\right\|_{A^*} \end{aligned}$$

holds. We can prove the "if" part in a similar calculation. Q.E.D.

Theorem 3. *A semisimple type I-BSE-algebra with a bounded approximate identity is isomorphic to a commutative* C^**-algebra and conversely.*

Proof. If A is isomorphic to a commutative C^*-algebra, then $A \cong C_0(\Phi_A)$ and hence $M(A)^\frown = C^b(\Phi_A)$. So by the preceding lemma, A is type I-BSE. Conversely let A be a semisimple type I-BSE-algebra with a bounded (say, by $\beta < \infty$) approximate identity $\{e_\lambda\}$. Define τ_x by the relation $\tau_x(y) = xy$ $(y \in A)$, then the set $I = \{\tau_x : x \in A\}$ is an ideal of $M(A)$ which is isomorphic to A. Note that $\|\tau_x\| \geq \|\tau_x(e_\lambda/\|e_\lambda\|)\| = (1/\|e_\lambda\|)\|xe_\lambda\| \geq (1/\beta)\|xe_\lambda\| \to (1/\beta)\|x\|$, so the operator norm on I is a complete norm. Therefore I is closed in $M(A)$. Also by the semisimplicity of A, the algebra homomorphism: $T \to T^\frown$ of $M(A)$ onto $M^\frown(A)$ is injective. Note also that $\|T^\frown\|_\infty \leq \beta\|T\|$ for all $T \in M(A)$. Since A is type I-BSE, $M^\frown(A) = C^b(\Phi_A)$, so $M(A) \cong C^b(\Phi_A)$ by the open mapping theorem. Consequently, $I^\frown = \{T^\frown : T \in I\}$ becomes a commutative C^*-algebra and $I^\frown \cong A$. Q.E.D.

Problem 1. Does Theorem 2 hold without the assumption of a bounded approximate identity?

Theorem 4. (i) $C_{\mathrm{BSE}}(\Phi_A)$ *equals the set of all* $\sigma \in C^b(\Phi_A)$ *for which there exists a bounded net* $\{x_\lambda\}$ *in* A *with* $\lim \hat{x}_\lambda(\varphi) = \sigma(\varphi)$ *for all* $\varphi \in \Phi_A$.

(ii) $C_{\mathrm{BSE}}(\Phi_A) = C^b(\Phi_A) \cap (A^{**}|\Phi_A)$, *where* A^{**} *denotes the second dual of* A.

Proof. (i) Let a function σ in $C^b(\Phi_A)$ be such that there exist $\beta < \infty$ and a net $\{x_\lambda\}$ in A with $\|x_\lambda\| \leq \beta$ for all λ and $\lim \hat{x}_\lambda(\varphi) = \sigma(\varphi)$ for all $\varphi \in \Phi_A$. To show $\sigma \in C_{\mathrm{BSE}}(\Phi_A)$, let $c_1, \dots, c_n$ be complex numbers in Δ and $\varphi_1, \dots, \varphi_n$ be points in Φ_A. Then we have

$$\left|\sum_{i=1}^n c_i \sigma(\varphi_i)\right| \leq \left|\sum_{i=1}^n c_i x_\lambda^\frown(\varphi_i)\right| + \left|\sum_{i=1}^n c_i \{x_\lambda^\frown(\varphi_i) - \sigma(\varphi_i)\}\right|$$
$$\leq \beta \left\|\sum_{i=1}^n c_i \varphi_i\right\|_{A^*} + \left|\sum_{i=1}^n c_i \{x_\lambda^\frown(\varphi_i) - \sigma(\varphi_i)\}\right|.$$

Taking the limit with respect to λ, we obtain $\sigma \in C_{\mathrm{BSE}}(\Phi_A)$. Conversely let $\sigma \in C_{\mathrm{BSE}}(\Phi_A)$. Let Λ be the net consisting of all finite subsets of Φ_A. Then by Helly's theorem, for each $\varepsilon > 0$ and $\lambda \in \Lambda$, there exists $x = x(\lambda, \varepsilon) \in A$ such that $\|x\| \leq \|\sigma\|_{\mathrm{BSE}} + \varepsilon$ and $x^\frown(\varphi) = \sigma(\varphi)$ for all $\varphi \in \lambda$. This shows that $\lim \hat{x}(\lambda, \varepsilon) = \sigma$ pointwise on Φ_A.

(ii) $C^b(\Phi_A) \cap (A^{**}|\Phi_A) \subset C_{\mathrm{BSE}}(\Phi_A)$ is straightforward. To show the reverse inclusion, let $\sigma \in C_{\mathrm{BSE}}(\Phi_A)$. Then by (i), there exist $\beta < \infty$ and a net $\{x_\lambda\}$ in A with $\|x_\lambda\| \leq \beta$ for all λ and $\lim \hat{x}_\lambda(\varphi) = \sigma(\varphi)$ for all $\varphi \in \Phi_A$. Let π be the natural embedding of A into A^{**}. Then there exists a subnet $\{x_{\lambda'}\}$ of $\{x_\lambda\}$ and $F \in A^{**}$ such that $w^*\text{-}\lim \pi(x_{\lambda'}) = F$. Therefore for each $\varphi \in \Phi_A$,

$$\sigma(\varphi) = \lim \hat{x}_{\lambda'}(\varphi) = \lim \pi(x_{\lambda'})(\varphi) = F(\varphi),$$

so that $\sigma = F|\Phi_A$. Q.E.D.

Remarks. By the above proof, for each $\sigma \in C_{\mathrm{BSE}}(\Phi_A)$,

$$\|\sigma\|_{\mathrm{BSE}} = \inf\{\beta > 0 \colon \exists \text{ a net } \{x_\lambda\} \subset A \text{ with } \|x_\lambda\| \le \beta \quad (\forall \lambda), \quad \lim \hat{x}_\lambda(\varphi) = \sigma(\varphi) \quad (\forall \varphi \in \Phi_A)\}.$$

Also the algebra $C_{\mathrm{BSE}}(\Phi_A)$ is closed in the following sense: If σ is a function in $C^b(\Phi_A)$ such that there exist $\beta < \infty$ and a net $\{\sigma_\lambda\}$ in $C_{\mathrm{BSE}}(\Phi_A)$ with $\|\sigma_\lambda\| \le \beta$ for every λ and $\lim \sigma_\lambda(\varphi) = \sigma(\varphi)$ for every $\varphi \in \Phi_A$, then σ must be in $C_{\mathrm{BSE}}(\Phi_A)$.

Definition. If $\{e_\lambda\}$ is a bounded net in A satisfying the conditions $\lim \varphi(xe_\lambda) = \varphi(x)$ for every $x \in A$ and $\varphi \in \Phi_A$, then we call it a bounded weak approximate identity of A in the sense of Jones–Lahr (cf. [2] or [6]).

Corollary 5. *$M^\frown(A) \subset C_{\mathrm{BSE}}(\Phi_A)$ if and only if A has a bounded weak approximate identity in the sense of Jones–Lahr.*

Proof. Since $M^\frown(A)$ contains the identity of $C^b(\Phi_A)$, the "only if" part follows immediately from Theorem 4(i). To show the "if" part, assume that A has a bounded (say, by $\beta < \infty$) weak approximate identity $\{e_\lambda\}$ in the sense of Jones–Lahr. Let T be a multiplier on A. Then for any complex numbers $c_1, \dots, c_n$ and the same number of $\varphi_1, \dots, \varphi_n$ in Φ_A,

$$\begin{aligned}\left|\sum_{i=1}^n c_i T^\frown(\varphi_i)\right| &\le \left|\sum_{i=1}^n c_i T^\frown(\varphi_i) e_\lambda^\frown(\varphi_i)\right| + \left|\sum_{i=1}^n c_i T^\frown(\varphi_i)\{1 - e_\lambda^\frown(\varphi_i)\}\right| \\ &\le \|Te_\lambda\| \left\|\sum_{i=1}^n c_i \varphi_i\right\|_{A^*} + \left|\sum_{i=1}^n c_i T^\frown(\varphi_i)\{1 - e_\lambda^\frown(\varphi_i)\}\right|. \\ &\le \beta\|T\| \left\|\sum_{i=1}^n c_i \varphi_i\right\|_{A^*} + \left|\sum_{i=1}^n c_i T^\frown(\varphi_i)\{1 - e_\lambda^\frown(\varphi_i)\}\right|.\end{aligned}$$

Hence, by taking the limit with respect to λ, we obtain $\|T^\frown\|_{\mathrm{BSE}} \le \beta\|T\|$ and hence $T^\frown \in C_{\mathrm{BSE}}(\Phi_A)$. Q.E.D.

Corollary 6. *Let A be a BSE-algebra. Then A is semisimple if and only if there exist positive real numbers α and β such that $\alpha\|T\| \le \|T^\frown\|_{\mathrm{BSE}} \le \beta\|T\|$ for all $T \in M(A)$.*

Proof. Suppose that A is semisimple. Then the map: $T \to T^\frown$ is an algebra isomorphism of $M(A)$ onto $C_{\mathrm{BSE}}(\Phi_A)$ $(= M^\frown(A))$. Also there exists $\beta < \infty$ such that $\|T^\frown\|_{\mathrm{BSE}} \le \beta\|T\|$ for all $T \in M(A)$ as observed in the proof of the preceding corollary. Therefore we can find $\alpha < \infty$ such that $\alpha\|T\| \le \|T^\frown\|_{\mathrm{BSE}}$ for all $T \in M(A)$ by the open mapping theorem. Suppose conversely that there exist positive real numbers α and β such that $\alpha\|T\| \le \|T^\frown\|_{\mathrm{BSE}} \le \beta\|T\|$ for all $T \in M(A)$. Let $x \in A$ be such that $x^\frown(\varphi) = 0$ for all $\varphi \in \Phi_A$. Note that $(\tau_x)^\frown = x^\frown = 0$, so $\tau_x = 0$ by the assumption. Then $x = 0$, since A has no nonzero annihilators. In other words, A is semisimple. Q.E.D.

Remark. The multiplier algebra of a semisimple BSE-algebra A is topologically isomorphic to $C_{\mathrm{BSE}}(\Phi_A)$ by the above corollary.

Theorem 7. *Both the disk algebra and the classical Hardy algebra are type II-BSE.*

Proof. Let A be either the disk algebra or the classical Hardy algebra and D the open unit disk. We show that $A = C_{\mathrm{BSE}}(\Phi_A)$. To do this let $f \in C_{\mathrm{BSE}}(\Phi_A)$. By Theorem 4(i), there exist $\beta < \infty$ and a net $\{f_\lambda\}$ in A with the index set Λ such that $\|f_\lambda\| \le \beta$ for every λ and $\lim f_\lambda(z) = f(z)$ for every $z \in D$. If we can show that

$$\lim \|f_\lambda - f\|_K = 0 \tag{1}$$

for all compact subset K of D, then we conclude by Morera's theorem that $f \in A$. Here $\|f\|_K = \sup\{|f(z)|: z \in K\}$. Thus it only remains to show the equality (1). It is trivial that in any compact Hausdorff space S a net which has at most one cluster point must in fact converge. Now consider the closure S of the set $\{f_\lambda: \lambda \in \Lambda\}$ in the compact-convergence topology (denoted by cc) of $H(D)$. Since S is bounded, it is cc-compact by Montel's theorem. Since cc-convergence implies pointwise convergence, every cc-cluster point of the net $\{f_\lambda\}$ cc-converges, necessarily to f. Q.E.D.

3. Ideals and quotient algebras of BSE-algebras

In this section we will consider the following problem: Are closed ideals and quotient algebras of BSE-algebras BSE? We will deal only with Banach algebras with discrete carrier space.

When a closed ideal I of a commutative Banach algebra A is essential as Banach A-module, that is, I equals the closed linear span of $\{ax: a \in A, x \in I\}$, we will call I an essential ideal.

Theorem 8. *Let A be a BSE-algebra with discrete carrier space and I an essential closed ideal of A. Then*

(i) $M\hat{\ }(A/I) = C_{\mathrm{BSE}}(\Phi_{A/I})$.
(ii) $C_{\mathrm{BSE}}(\Phi_I) \subset M\hat{\ }(I)$.

In particular, if the quotient $I : A$ of I (cf. [9] *for definition) is contained in I, then A/I is BSE. Also if I has a bounded weak approximate identity in the sense of Jones–Lahr and has no nonzero annihilators, then I is BSE.*

Proof. (i) Since A is BSE, it has a bounded weak approximate identity in the sense of Jones–Lahr by Corollary 5. Then A/I also has an approximate identity in the sense of Jones–Lahr. Hence, by Corollary 5, we have that $M\hat{\ }(A/I) \subset C_{\mathrm{BSE}}(\Phi_{A/I})$. To show the reverse inclusion, let $\sigma' \in C_{\mathrm{BSE}}(\Phi_{A/I})$. Since both Φ_A and $\Phi_{A/I}$ are discrete, it follows from Theorem 4(ii) that

$$C_{\mathrm{BSE}}(\Phi_A) = A^{**}|\Phi_A \tag{2}$$

and

$$C_{\mathrm{BSE}}(\Phi_{A/I}) = (A/I)^{**}|\Phi_{A/I}. \tag{3}$$

By (3) we can take $F' \in (A/I)^{**}$ with $F'|\Phi_{A/I} = \sigma'$. Set $I^\perp = \{f \in A^* : f|I = 0\}$. For each $f \in I^\perp$, define f^* by the relation $f'(x') = f(x)$ $(x' = x + I \in A/I)$. The map: $f \to f'$ is an isometric isomorphism of $I^\perp$ onto $(A/I)^*$. We define $\widetilde{F'}$ by the relation $\widetilde{F'}(f) = F'(f')$ $(f \in I^\perp)$. Then $\widetilde{F'} \in (I^\perp)^*$ and hence there is $F \in A^{**}$ such that $\|F\| = \|\widetilde{F'}\|$ and $F|I^\perp = \widetilde{F'}$ from the Hahn–Banach extension theorem. By (2) and the BSE property of A, there exists $T \in M(A)$ with $T^\frown = F|\Phi_A$. Since I is essential and $T(xy) = xTy$ for all x, $y \in A$ (for A has no nonzero annihilators), we see $T(I) \subset I$ and hence T' defined by $T'(x') = (Tx)'$ $(x \in A)$ belongs to $M(A/I)$. In this case $T'^\frown = \sigma'$. Actually for any $x \in A$ and $\varphi \in \Phi_A \cap I^\perp$,

$$\begin{aligned} x^\frown(\varphi)T'^\frown(\varphi') &= x'^\frown(\varphi')T'^\frown(\varphi') = (T'x')^\frown(\varphi') = (Tx)'^\frown(\varphi') \\ &= (Tx)^\frown(\varphi) = x^\frown(\varphi)T^\frown(\varphi), \end{aligned}$$

so that $T'^\frown = T^\frown(\varphi)$ and hence

$$\sigma'(\varphi') = F'(\varphi') = \widetilde{F'}(\varphi) = F(\varphi) = T^\frown(\varphi) = T'^\frown(\varphi').$$

Therefore $\sigma' = T'^\frown$, since $\Phi_{A/I} = \{\varphi' : \varphi \in \Phi_A \cap I^\perp\}$. We thus conclude that $\sigma' \in M^\frown(A/I)$. In other words, $M^\frown(A/I) \supset C_{\mathrm{BSE}}(\Phi_{A/I})$.

(ii) Let $\omega \in C_{\mathrm{BSE}}(\Phi_I)$. Note that $\Phi_I = \{\varphi|I : \varphi \in \Phi_A \backslash I^\perp\}$, which is also discrete (see [9, Theorem 3.1.18]). So a complex-valued function σ on Φ_A defined by

$$\sigma(\varphi) = \begin{cases} \omega(\varphi|I) & (\varphi \in \Phi_A \backslash I^\perp) \\ 0 & (\varphi \in \Phi_A \cap I^\perp) \end{cases}$$

belongs to $C_{\mathrm{BSE}}(\Phi_A)$. In fact for every finite number of complex numbers $c_1, \ldots, c_n$ and the same number of $\varphi_1, \ldots, \varphi_n$ in Φ_A, using $\sum'$ to denote the summation over $\varphi_i \in \Phi_A \backslash I^\perp$,

$$\begin{aligned} \left|\sum_{i=1}^n c_i\sigma(\varphi_i)\right| &= \left|\sum{}' c_i\sigma(\varphi_i)\right| = \left|\sum{}' c_i\omega(\varphi_i|I)\right| \\ &\le \|\omega\|_{\mathrm{BSE}} \left\|\sum{}' c_i\varphi_i|I\right\|_{I^*} \\ &= \|\omega\|_{\mathrm{BSE}} \left\|\sum_{i=1}^n c_i\varphi_i|I\right\|_{I^*} \\ &\le \|\omega\|_{\mathrm{BSE}} \left\|\sum_{i=1}^n c_i\varphi_i\right\|_{A^*}, \end{aligned}$$

so that $\sigma \in C_{\mathrm{BSE}}(\Phi_A)$. As A is BSE, take $T \in M(A)$ such that $T^\frown = \sigma$ and put $S = T|I$. Then $S \in M(I)$. Also for any $x \in I$ and $\varphi \in \Phi_A \backslash I^\perp$, we have

$$\begin{aligned} S^\frown(\varphi|I)x^\frown(\varphi|I) &= (Sx)^\frown(\varphi|I) = (Tx)^\frown(\varphi) = T^\frown(\varphi)x^\frown(\varphi) \\ &= \sigma(\varphi)\hat{x}(\varphi|I) = \omega(\varphi|I)x^\frown(\varphi|I). \end{aligned}$$

Then $\omega = S^{\wedge} \in M^{\wedge}(I)$. In other words, $C_{\mathrm{BSE}}(\Phi_I) \subset M^{\wedge}(I)$.

In particular, if $I: A \subset I$ then A/I has no nonzero annihilators and hence it is BSE by (i). Also if I has a bounded weak approximate identity in the sense of Jones–Lahr, then $C_{\mathrm{BSE}}(\Phi_1) = M^{\wedge}(\Phi_I)$ by (ii) and Corollary 5. Therefore if further I has no nonzero annihilators, then I is BSE. Q.E.D.

Remark. Let A be a commutative Banach algebra and I a closed ideal of A. Then $I: A \subset I$ either if A has an approximate identity or if I is modular. If also A has a bounded weak approximate identity in the sense of Jones–Lahr and I is kernel then $I: A \subset I$.

Corollary 9. *Every quotient algebra of the group algebra on a compact Abelian group is BSE.*

Remark. We don't know if a closed ideal of the group algebra of an arbitrary compact Abelian group has a bounded weak approximate identity in the sense of Jones–Lahr.

4. Counterexamples

Let l^1 be the commutative Banach algebra of all absolutely convergent sequences of complex numbers with pointwise multiplication. Then

$$l^{1\wedge} = C_{\mathrm{BSE}}(\Phi_{l^1}) \subsetneqq M^{\wedge}(l^1) = C^b(\Phi_{l^1}).$$

Therefore l^1 is not BSE and does not have a bounded weak approximate identity in the sense of Jones–Lahr. Let $\mathbf{N}$ be the semigroup of all natural numbers and $L^1(\mathbf{N})$ be the semigroup algebra of $\mathbf{N}$. We can show that $M^{\wedge}(L^1(\mathbf{N}))$ is not contained in $C_{\mathrm{BSE}}(\Phi_{L^1(\mathbf{N})})$. So $L^1(\mathbf{N})$ is also not BSE and does not have a bounded weak approximate identity in the sense of Jones-Lahr. However the semigroup algebra on the semigroup of all positive real numbers possesses a bounded weak approximate identity in the sense of Jones–Lahr (cf. [6]). In spite of this fact we don't know whether this algebra is BSE or not.

We conclude this section with the following remark pointed out by K. Izuchi: The measure algebra $M(G)$ of a nondiscrete locally compact Abelian group G is not BSE. In fact, let $G^{\wedge}$ be the dual group of G and choose a measure μ in $M(G)$ such that its Fourier–Stieltjes transform $\mu^{\wedge}$ vanishes at infinity and its Gelfand transform $\mu^{\vee}$ is not identically zero on $\Phi_{M(G)} \backslash G^{\wedge}$ (cf. [1, 3 or 4]). Next define f by $f = \mu^{\wedge}$ on $G^{\wedge}$ and $f = 0$ on $\Phi_{M(G)} \backslash G^{\wedge}$. We see easily that f belongs to $C_{\mathrm{BSE}}(\Phi_{M(G)})$ but there does not exist a measure ν in $M(G)$ with $\nu^{\vee} = f$.

In [5] K. Izuchi gives many important examples and counterexamples of BSE-algebras.

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