

## ON THE TOPOLOGY OF THE SPACE OF CONVOLUTION OPERATORS IN $K'_M$

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**ABSTRACT.** In this paper we show that on the space  $O'_c(K'_M: K'_M)$  of convolution operators on  $K'_M$ , the topology  $\tau_b$  of uniform convergence on bounded subsets of  $K'_M$  is equal to the strong dual topology.

### INTRODUCTION

In previous work (see [1, 2]) we redefined the space  $K'_M$  of rapidly increasing distributions and the space  $O'_c(K'_M: K'_M)$  of its convolution operators. The space  $O'_c(K'_M: K'_M)$  was provided with several topologies, the topology  $\tau_b$  of uniform convergence on bounded subsets of  $K'_M$ , the topology of  $\tau'_b$  of uniform convergence on bounded subsets  $K'_M$ , the projective limit topology  $\tau_p$  where  $O'_c(K'_M: K'_M)$  was considered as the projective limit of the spaces  $e^{-M(kx)}S'$ , and the strong dual topology where  $O'_c(K'_M: K'_M)$  was considered as the dual of  $O_c(K_M: K_M)$  (see definitions below). It was shown in [2] that  $\tau_b$  and  $\tau'_b$  are equal, and  $\tau_p$  is equal to the strong dual topology. The question whether  $\tau_b$  and the strong dual topology are equal or not was left unanswered in [2]. Our main result in this paper is the following.

**Theorem.** *On the space  $O'_c(K'_M: K'_M)$  the topology  $\tau_b$  is equal to the strong dual topology.*

To establish this result we need the following

**Lemma.** *The convolution map from  $(O'_c, \tau_b) \times O_c$  into  $O_c$  is separately continuous.*

To avoid lengthy proofs we will present these results in several steps.

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## NOTATIONS AND PRELIMINARY RESULTS

By  $N^n$ ,  $R^n$  we denote the sets of  $n$ -tuples of nonnegative integers and real numbers, respectively. For  $\alpha = (\alpha_1, \dots, \alpha_n)$  in  $N^n$  we denote by  $|\alpha|$  the sum  $\alpha_1 + \dots + \alpha_n$ . By  $D$  and  $D'$  we denote the Schwartz spaces of test functions and distributions, by  $S$  we denote the space of infinitely differentiable functions rapidly decreasing at infinity and its strong dual  $S'$  is the space of tempered distribution. For any distribution  $T$  we denote by  $\check{T}$  its symmetry with respect to the origin and by  $\tau_h T$ ,  $h \in R^n$ , the translation of  $T$  by  $h$ . For  $\alpha \in N^n$  we denote by  $D$  the differential operator  $D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n}$ ; where

$$D_j = \frac{1}{i} \frac{\partial}{\partial x_j}; \quad j = 1, 2, \dots, n.$$

Let  $E$  be a locally convex topological vector space and  $E'$  its strong dual, for a bounded subset  $B$  of  $E$  we denote by  $B^\circ$  the polar of  $B$ , which is the set of all  $T$  in  $E'$  such that  $|\langle T, \phi \rangle| < 1$  for all  $\phi$  in  $B$ . We will use the results of our previous work [1] and [2].

Let  $\mu(\xi)$ ;  $0 \leq \xi \leq \infty$  be a nonnegative increasing function with  $\mu(0) = 0$ ,  $\mu(\infty) = \infty$ . For any  $x$ ,  $0 \leq x < \infty$  define the function  $M(x) = \int_0^x \mu(\xi) d\xi$ . It follows that  $M$  is positive, increasing, and convex. For negative  $x$  we define  $M$  by symmetry, i.e.  $M(x) = M(-x)$ . For  $x = (x_1, x_2, \dots, x_n)$  we define  $M(x) = M(x_1) + M(x_2) + \dots + M(x_n)$ . The function  $M$  satisfies the inequality  $M(x+y) \leq M(2x) + M(2y)$  for all  $x, y$  in  $R^n$ . We will write  $w^k$  and  $w^{-k}$  for the function  $\exp[M(kx)]$  and  $\exp[-M(kx)]$ , respectively;  $k = 0, 1, 2, \dots$

The space  $K_M$  consists of all  $C^\infty$ -functions  $\phi$  such that

$$\nu_k(\phi) = \sup_{\substack{x \in R^n \\ |\alpha| \leq k}} w^k |D^\alpha \phi(x)|; \quad k = 0, 1, 2, \dots$$

It has been proved in Abdullah [1] that  $K_M$  is a Fréchet nuclear space, moreover it is Montel (hence reflexive), bornologic and is a normal space of distributions. By  $K'_M$  we denote the strong dual of  $K_M$  provided with the topology of uniform convergence on bounded subsets of  $K_M$ ,  $K'_M$  is the space of distributions which grow not faster than  $\exp(M(kx))$  for some  $k \geq 0$ . The elements of  $K'_M$  are called distributions of rapid growth. In the case  $M(x) = |x|^p/p$ ;  $p > 1$ , the spaces  $K_M$ ,  $K'_M$  are the spaces  $K_p$  and  $K'_p$  of Sampson and Zielezny [4]. It turns out that  $K'_M$  is bornologic. For  $T \in K'_M$  and  $\phi \in K_M$  we define the convolution of  $T$  and  $\phi$  by the relation  $(T * \phi)(x) = \langle T_y, \phi(x-y) \rangle$ .

The following theorem (see [2], Theorem 1) will be used later in the proofs; its proof is similar to the proof of the corresponding result for the special case  $M(x) = |x|^p/p$ ,  $p > 1$  and will be omitted (see Theorem 2 of [4]).

**Theorem A.** *Let  $S$  be any element of  $K'_M$ , the following statements are equivalent.*

- (a)  $S$  is in  $O'_c(K'_M; K'_M)$ .

- (b) The distribution  $w^k S$ ,  $k = 0, 1, 2, \dots$ , are in  $S'$ .
- (c) For any  $k \geq 0$  there exists a nonnegative integer  $m$  such that

$$S = \sum_{|\alpha| \leq m} D^\alpha f_\alpha,$$

where, for each  $\alpha$ ,  $f_\alpha$  is a continuous function such that  $w^k f_\alpha \in L^\infty$ .

- (d) For any  $k$ , the set of distributions  $\{w^{-k}(h)\tau_h S : h \in \mathbb{R}^n\}$  is bounded in  $D'$ .
- (e)  $S * \phi$  is in  $K_M$  for all  $\phi$  in  $K_M$  and the map  $\phi \rightarrow S * \phi$  from  $K_M$  into  $K_M$  is continuous.

For  $S \in O'_c(K'_M : K'_M)$  and  $T \in K'_M$  we define  $S * T$ , the convolution of  $S$  and  $T$ , by  $\langle S * T, \phi \rangle = \langle T, \check{S} * \phi \rangle$ ;  $\phi \in K_M$ , where  $(\check{S} * \phi)(x) = \langle S_y, \check{\phi}(x - y) \rangle$ . Let  $(T_j)$  be a sequence in  $K'_M$  converging to 0; from property (3) of Theorem 5 of [1] it follows that  $S * B$  is bounded in  $K_M$  for every bounded subset  $B$  of  $K_M$ . Hence  $\langle S * T_j, \phi \rangle = \langle T_j, \check{S} * \phi \rangle \rightarrow 0$  uniformly in  $\phi \in B$ . Since  $K'_M$  is bornological it follows that the map  $T \rightarrow S * T$  from  $K'_M$  into  $K'_M$  is continuous. The space  $O'_c(K'_M : K'_M)$  is the space of convolution operators in  $K'_M$ .

The space  $O'_c(K'_M : K'_M)$  will be provided with several topologies. The first topology  $\tau_p$  is the projective limit topology of the spaces  $w^{-k}S'$ , a second topology  $\tau_b$  induced by  $L_b(K_M : K_M)$ , the space of all continuous linear maps from  $K_M$  into itself provided with the topology of uniform convergence on bounded subsets of  $K_M$ , and a third topology  $\tau'_b$  is induced by  $L_b(K'_M : K'_M)$ , the space of all continuous linear maps from  $K'_M$  into itself provided with the topology of uniform convergence on bounded subsets of  $K'_M$ .

We denote by  $O_c(K'_M : K'_M)$  the union of the spaces  $w^k S$ ,  $k = 0, 1, 2, \dots$ , provided with the inductive limit topology. Since  $S$  is bornological (being metrizable), and the inductive limit of bornological spaces is bornological, it follows that  $O_c(K'_M : K'_M)$  is bornological. The space  $O_c(K'_M : K'_M)$  is the strong dual of  $O'_c(K'_M : K'_M)$  with the topology  $\tau_p$ , (see [2], Theorem 3). Moreover, on  $O_c(K'_M : K'_M)$  the strong dual topology, i.e. the topology of uniform convergence on bounded subsets of  $O'_c(K'_M : K'_M)$ , is equivalent to the inductive limit topology (see [2], Theorem 5).

On  $O'_c(K'_M : K'_M)$ , the topologies  $\tau_b$  and  $\tau'_b$  are equivalent (see [1], Theorem 6), and with  $\tau_b$  the space  $O'_c(K'_M : K'_M)$  is complete and nuclear. Moreover,  $O'_c(K'_M : K'_M)$  with  $\tau_b$  is bornological, the proof is similar to the space  $O'_c(K'_1 : K'_1)$  of [6]. Another topology on  $O'_c(K'_M : K'_M)$  is the strong dual topology, which is the topology of uniform convergence on bounded subsets of  $O_c(K'_M : K'_M)$ . We denote by  $(O'_c, \text{sdt})$  the space  $O'_c(K'_M : K'_M)$  provided with the strong dual topology. The strong dual topology of  $O'_c(K'_M : K'_M)$  is equiva-

lent to the projective limit topology  $\tau_p$  (see [2], Theorem 4). From now on, we will write  $O_c, O'_c$  for  $O_c(K'_M:K'_M)$  and  $O'_c(K'_M:K'_M)$ , respectively.

### THE RESULTS

We begin with a preliminary result that will be used in the proofs of the subsequent results.

**Theorem 1.** *For any  $T$  in  $K'_M$  and  $\phi$  in  $K_M$  the convolution  $T * \phi$  is in  $O_c$ , moreover the bilinear map  $\Lambda$  from  $K'_M \times K_M$  into  $O_c$  mapping  $(T, \phi)$  to  $T * \phi$  is separately continuous in both variables.*

*Proof.* Let  $T$  be any element in  $K'_M$ , by the characterization theorem of elements of  $K'_M$  it follows that there exist a multi-index  $\alpha$ , a positive integer  $k$ , and a bounded continuous function  $f$  such that  $T = D^\alpha[w^k f(x)]$ . For any multi-index  $\beta$ , from the properties of  $M$ , one has

$$(1) \quad \begin{aligned} |D^\beta(T * \phi)(x)| &= \left| (-1)^{|\alpha|} \int w^k(y) f(y) D^{(\alpha+\beta)}\phi(x-y) dy \right|; \\ &\leq c_1 w^{2k}(x) \int w^{2k}(x-y) |D^{(\alpha+\beta)}\phi(x-y)| dy; \\ &\leq c w^{2k}(x), \end{aligned}$$

where  $k, c$  are constants and  $k$  is independent of  $\beta$ . Hence  $T * \phi \in O_c$ . Next, we prove continuity of  $\Lambda$  in the second variable. For fixed  $T$  in  $K'_M$  let  $\Lambda_T$  be the linear map from  $K_M$  into  $O_c$  taking  $\phi$  to  $T * \phi$ . Since  $K_M$  is bornological, from Theorem II.8.3 of Schaefer [5] it suffices to show that  $\Lambda_T$  is sequentially continuous. Let  $(\phi_j)$  be a sequence in  $K_M$  converging to 0, from inequality (1) it follows that the set  $\{T * \phi_j; j = 1, 2, \dots\}$  is contained in  $w^{4k}S$ , for that fixed  $k$ . Moreover, for any polynomial  $p(x)$  and any positive integer  $m$  one has

$$(2) \quad \begin{aligned} \sup_{\substack{x \in R^n \\ |\beta| \leq m}} |p(x)| |D^\beta(w^{-4k}(x)(T * \phi_j)(x))| \\ \leq c_2 c_k \sup_{\substack{x \in R^n \\ |\beta| \leq m \\ \alpha \leq \beta}} |p(x)| w^{-4k}(x) w^{2k}(x) \int w^{2k}(y) |D^{\alpha+\beta}\phi_j(y)| dy. \end{aligned}$$

The convergence of  $(\phi_j)$  to 0 in  $K_M$  implies that the integral converges to 0 as  $j$  goes to infinity. Since  $\sup_{x \in R^n} |p(x)| w^{-2k}(x) < \infty$ , inequality (2) implies that  $(w^{-4k}T * \phi_j)$  converges to 0 in  $S$ , hence  $(T * \phi_j)$  converges to 0 in  $w^{4k}S$ . Thus  $(T * \phi_j)$  converges to 0 in  $O_c$ . This establishes the continuity of  $\Lambda_T$ .

Finally, we show continuity of  $\Lambda$  in the first variable. For fixed  $\phi \in K_M$  let  $\Lambda_\phi$  be the linear map from  $K'_M$  into  $O_c$  taking  $T$  to  $T * \phi$ ; since  $K'_M$  is bornological it suffices to show that  $\Lambda_\phi$  is sequentially continuous. Let  $(T_j)$

be a sequence in  $K'_M$  converging to 0, we show  $\Lambda_\phi(T_j) = T_j * \phi \rightarrow 0$  in  $O_c$ . We claim that  $\Lambda_\phi$  from  $O'_c$  into  $K_M$  is continuous. For, given  $U$  any neighborhood of 0 in  $K_M$ , consider the set  $V(\{\phi\}, U) = \{S \in O'_c : \phi * S \in U\}$ ,  $V(\{\phi\}, U)$  is a member of 0-neighborhood base in  $(O'_c, \tau_b)$ , and  $\Lambda_\phi^{-1}(U) = V(\{\phi\}, U)$ , this proves the claim. Let  $B'$  be any bounded subset of  $O'_c$ , the above assertion implies that  $\Lambda_{\check{\phi}}(B') = \{\check{\phi} * S : S \in B'\}$  is bounded in  $K_M$ . Hence

$$\langle \Lambda_\phi(T_j), S \rangle = \langle T_j, \check{\phi} * S \rangle \rightarrow 0 \text{ uniformly in } S \in B',$$

i.e.  $\Lambda_\phi(T_j)$  converges to 0 in  $O_c$ . This completes the proof of the theorem.

The next theorem represents one direction of the main result in the paper, before we give the proof of the other half we need a few lemmas.

**Theorem 2.** *On the space  $O'_c$  the topology  $\tau_b$  is weaker than the strong dual topology.*

*Proof.* Let

$$V(B, U) = \{S \in O'_c : S * \phi \in U \text{ for all } \phi \text{ in } B\};$$

where  $U$  is a neighborhood of 0 in  $K_M$  and  $B$  is a bounded subset of  $K_M$ , be a member of a 0-neighborhood base of  $\tau_b$ . Since  $K_M$  is reflexive we can assume without loss of generality that  $U = (B')^\circ$ , the polar of  $B'$  a bounded subset of  $K'_M$ . Also, we can assume that  $B = \check{B}$  and  $B' = \check{B}' = \{\check{T} : T \in B'\}$ . For every  $T$  in  $B'$  consider the linear map  $\Lambda_T$  from  $K_M$  into  $O_c$  taking  $\phi$  to  $T * \phi$ , set  $\Gamma = \{\Lambda_T : T \in B'\}$ . From Theorem 1 it follows that for each  $T$  in  $B'$  and each  $\phi$  in  $K_M$  the linear maps  $\Lambda_T$  and  $\Lambda_\phi$  are continuous, hence for each  $\phi$  in  $K_M$  the orbit  $\Gamma(\phi) = \Lambda_\phi(B') = \{T * \phi : T \in B'\}$  is bounded in  $O_c$ . Hence, the set of all  $\phi$  in  $K_M$  such that  $\Gamma(\phi) = \{\Lambda_T(\phi) = T * \phi : T \in B'\}$  is bounded in  $O_c$  consists of all of  $K_M$ . Since  $K_M$  is a complete metric space it is of second category. From the Banach–Steinhaus theorem (see [3], Theorem 2.5, p. 43) it follows that  $\Gamma$  is an equicontinuous family. By Theorem 2.4 of [3] there exists a bounded subset  $B_c$  of  $O_c$  such that  $\Lambda_T(B) \subset B_c$  for each  $\Lambda_T$  in  $\Gamma$ , i.e.  $T * B$  is contained in  $B_c$  for each  $T$  in  $B'$ . Hence,  $B' * B = \bigcup_{T \in B'} T * B \subset B_c$ . Being a subset of a bounded subset of  $O_c$  it follows that  $B * B'$  is bounded in  $O_c$ . To complete the proof of the assertion it suffices to show that  $V(B, U) = (B' * B)^\circ$ . For this, let  $S \in (B' * B)^\circ$ , one has  $|\langle S * \phi, T \rangle| = |\langle S, \check{\phi} * T \rangle| < 1$ , for all  $\phi$  in  $B$  and all  $T$  in  $B'$ , i.e.  $S \in V(B, U)$ . Conversely, if  $S \in V(B, U)$  then  $S * \phi \in (B')^\circ$  for all  $\phi$  in  $B$ , i.e.  $|\langle S * \phi, T \rangle| = |\langle S, \check{\phi} * T \rangle| < 1$ , for all  $\phi$  in  $B$  and  $T$  in  $B'$ . Hence  $S \in (B' * B)^\circ$ .

**Lemma 1.** *Let  $S$  be any element of  $O'_c$ . The convolution map  $\Lambda_S$  from  $O_c$  into itself which maps  $\psi$  to  $S * \psi$  is continuous.*

*Proof.* First we show that  $S * \psi$  is in  $O_c$  whenever  $\psi$  is in  $O_c$ . Since  $O_c = \bigcup_{k=0}^\infty w^k S$  it follows that  $\psi = w^k \phi$ , for some  $\phi$  in  $S$  and nonnegative integer

$k$ . From the characterization theorem of  $O'_c$  (Theorem A) it follows that there exists a nonnegative integer  $m = m(k)$  such that  $S = \sum_{|\alpha| \leq m} D^\alpha f_\alpha$ ; where the  $f_\alpha$ 's are functions with  $w^{2k} f_\alpha$  in  $L^\infty$ . Thus for any multi-index  $\beta$  one has

$$\begin{aligned}
 |D^\beta (S * \psi)(x)| &= |S * (D^\beta \psi)(x)| = \left| \sum_{|\alpha| \leq m} \langle D^\alpha f_\alpha(y), D^\beta \psi(x-y) \rangle \right|; \\
 &= \left| \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \langle f_\alpha(y), D^{\alpha+\beta} \psi(x-y) \rangle \right|; \\
 &= \left| \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \int f_\alpha(y) D^{\alpha+\beta} \psi(x-y) dy \right|; \\
 (3) \quad &\leq \sum_{|\alpha| \leq m} \int |f_\alpha(y)| |D^{\alpha+\beta} \psi(x-y)| dy; \\
 &\leq \sum_{|\alpha| \leq m} c_1 \int |f_\alpha(y)| w^{2k}(x-y) |\phi_{\alpha,\beta}(x-y)| dy; \\
 &\leq c_1 w^{4k}(x) \sum_{|\alpha| \leq m} \int w^{4k}(y) |f_\alpha(y)| |\phi_{\alpha,\beta}(x-y)| dy; \\
 &\leq c_1 c_2 w^{4k}(x) \sum_{|\alpha| \leq m} \int |\phi_{\alpha,\beta}(x-y)| dy; \\
 &\leq c_1 c_2 c_3 w^{4k}(x) = c w^{4k}(x),
 \end{aligned}$$

where  $\phi_{\alpha,\beta}$  is an element of  $S$  which depends on  $\psi$ ,  $\alpha$  and  $\beta$ , and  $c$  is a constant which depends on  $\alpha$ ,  $\beta$  and  $k$ , and  $k$  is independent of  $\beta$ . Hence  $S * \psi$  is in  $O_c$ .

Next we show the continuity of  $\Lambda_S$ . Since  $O_c$  is bornological it suffices to show that it is sequentially continuous. Let  $(\psi_j)$  be a sequence converging to 0 in  $O_c$ , we show that  $(S * \psi_j)$  converges to 0 in  $O_c$ . We consider  $O_c$  with the inductive limit topology. There exists a fixed  $k_1 \geq 0$  such that  $(\psi_j)$  converges to 0 in  $\omega^{k_1} S$ . Once more, using the characterization theorem of the space  $O'_c$  and the properties of  $M(x)$  one gets

$$\begin{aligned}
 w^{-k_1}(x) |(S * \psi_j)(x)| &\leq \sum_{|\alpha| \leq m} C \sup_{x,y \in \mathbb{R}^n} w^{-k_1}(x-y) |D^\alpha \psi_j(x-y)| \\
 &\quad \times \int w^{-k_1}(y) |f_\alpha(y)| dy; \\
 (4) \quad &\leq C \sup_{z \in \mathbb{R}^n} w^{-k_1}(z) |D^\alpha \psi_j(z)| \int w^{-k_1}(y) dy; \\
 &\leq C \sup_{z \in \mathbb{R}^n} w^{-k_1}(z) |D^\alpha \psi_j(z)|,
 \end{aligned}$$

where  $C$  is a constant which is not the same in all the estimates. Inequality (4) implies that  $(S * \psi_j)$  is contained in  $w^{2k_1}S$  and converges to 0 there (since the right-hand side of (4) converges to 0 in  $S$ ), hence  $(S * \psi_j)$  converges to 0 in  $O_c$ .

**Lemma 2.** *Let  $S$  be any element of  $O'_c$ . The linear map  $\Lambda_S$  from  $O'_c$  into itself taking  $T$  to  $S * T$  is continuous, where  $O'_c$  is provided with the strong dual topology.*

*Proof.* Let  $B^\circ$ , where  $B$  is a bounded subset of  $O_c$ , be a member of 0-neighborhood base of  $O'_c$ . One has

$$\begin{aligned} \Lambda_S^{-1}(B^\circ) &= \{T \in O'_c : S * T \in B^\circ\} \\ &= \{T \in O'_c : |\langle S * T, \psi \rangle| < 1 \text{ for all } \psi \text{ in } B\} \\ &= \{T \in O'_c : |\langle T, \check{S} * \psi \rangle| < 1 \text{ for all } \psi \text{ in } B\} \\ &= (\check{S} * B)^\circ. \end{aligned}$$

From Lemma 1 it follows that  $\check{S} * B$  is bounded in  $O_c$ , hence  $(\check{S} * B)^\circ$  is a member of 0-neighborhood base of  $O'_c$ . Thus  $\Lambda_S$  is continuous.

**Lemma 3.** *Let  $\psi$  be any element of  $O_c$ . The linear map  $\Lambda_\psi$  from  $(O'_c, \tau_b)$  into  $O_c$  is continuous.*

*Proof.* Since  $(O'_c, \tau_b)$  is bornological space it suffices to show that the map is bounded. Given  $B_c$  a bounded subset of  $(O'_c, \tau_b)$ , we show that  $\psi * B_c$  is bounded in  $O_c$ . Let  $V$  be a member of 0-neighborhood base in  $O_c$ , without loss of generality we can assume that  $V = B_1^\circ$ , where  $B_1$  is a bounded subset of  $(O'_c, \text{sdt})$ , that is,  $O'_c$  equipped with the strong dual topology. We find  $\lambda_1 > 0$  such that  $\lambda_1(\psi * B_c) \subset B_1^\circ$ . Since  $K_M$  is dense in  $O_c$  there exists a sequence  $(\phi_i)$  in  $K_M$  which converges to  $\psi$  in  $O_c$ . Thus for any  $W$ -neighborhood of 0 in  $O_c$  there exists a positive integer  $N(W)$  such that  $\phi_i \in W + \psi$  whenever  $i > N(W)$ . Let  $\phi = \phi_{2N(W)}$ , and consider the following member of 0-neighborhood base in  $(O'_c, \tau_b)$

$$M(\{\phi\}, B_1^\circ) = \left\{ S \in O'_c : S * \phi \in B_1^\circ \right\} = \left\{ S \in O'_c : |\langle S * \phi, T \rangle| < 1 \text{ for all } T \in B_1 \right\}.$$

Because  $B_c$  is bounded in  $(O'_c, \tau_b)$  there exists  $\lambda > 0$  such that  $\lambda B_c \subset M(\{\phi\}, B_1^\circ)$ , i.e.  $\lambda(\phi * B_c) \subset B_1^\circ$ .

Let  $W = (1/2\lambda)(\check{S} * B_1)^\circ$ , from Lemma 2 it follows that  $\check{S} * B_1$  is a bounded subset of  $(O'_c, \text{sdt})$ , hence  $W$  is a member of 0-neighborhood base for  $O_c$ .

Let  $\lambda_1 = \lambda/2$ , then for any  $S$  in  $B_c$  one has

$$\begin{aligned}
 (5) \quad |\langle \lambda_1(\psi * S), T \rangle| &= \lambda_1 \left| \langle \psi, \check{S} * T \rangle \right| \\
 &\leq \lambda_1 \left| \langle \phi, \check{S} * T \rangle \right| + \lambda_1 \left| \langle \psi - \phi, \check{S} * T \rangle \right| \\
 &= \frac{1}{2} |\langle \lambda(\phi * S), T \rangle| + \frac{1}{2} \lambda \left| \langle \psi - \phi, \check{S} * T \rangle \right| \\
 &< \frac{1}{2} \cdot 1 + \frac{1}{2} \lambda \cdot \frac{1}{2\lambda} = \frac{3}{4} < 1,
 \end{aligned}$$

for all  $T$  in  $B_1$ . Thus  $\lambda_1(\psi * B_c) \subset B_1^\circ$ , i.e.  $\psi * B_c$  is a bounded subset of  $O_c$ .

The next theorem establishes the remaining portion of our main result, in the proof the above lemmas will be used in an essential way.

**Theorem 3.** *On the space  $O'_c$ , the topology  $\tau'_b$  is not weaker than the strong dual topology.*

*Proof.* This will follow provided we show the continuity of the identity map from  $(O'_c, \tau'_b)$  into  $(O'_c, \text{sdt})$ . Since  $(O'_c, \tau'_b)$  is bornological it suffices to show that the map is sequentially continuous. Let  $(S_j)$  be a sequence converging to 0 in  $(O'_c, \tau'_b)$ , since  $O'_c$  with the strong dual topology is the dual of the Montel space  $O_c$ , we only need to show that  $(S_j)$  converges weakly in  $(O'_c, \text{sdt})$ . We consider first the case  $\psi$  in  $K_M$ . Since  $S_j * \psi \rightarrow 0$  in  $K_M$  it follows that  $\langle S_j, \psi \rangle = (S_j * \psi)(0) \rightarrow 0$ , i.e. the assertion is true for  $\psi$  in  $K_M$ . For general  $\psi$  in  $O_c$ , there exists a sequence  $(\psi_j)$  in  $K_M$  such that  $\psi_j$  converges to  $\psi$  in  $O_c$ . Let  $(\phi_\epsilon : \epsilon > 0)$  be a regularizing sequence in  $D$ , i.e.  $\phi_\epsilon \rightarrow \delta$  in  $E'$  as  $\epsilon \rightarrow 0$ . Thus one has

$$\lim_{\epsilon \rightarrow 0} \langle \check{S}_j * \psi_i, \check{\phi}_\epsilon \rangle = \lim_{\epsilon \rightarrow 0} \langle S_j, \phi_\epsilon * \psi_i \rangle = \langle S_j, \psi_i \rangle,$$

and by continuity of  $S_j$  one can define  $\langle S_j, \psi \rangle$  as follows:

$$\begin{aligned}
 (6) \quad \langle S_j, \psi \rangle &= \lim_{i \rightarrow \infty} \langle S_j, \psi_i \rangle = \lim_{i \rightarrow \infty} (S_j * \check{\psi}_i)(0) \\
 &= \lim_{i \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \langle \check{S}_j * \psi_i, \check{\phi}_\epsilon \rangle.
 \end{aligned}$$

Hence

$$(7) \quad \lim_{j \rightarrow \infty} \langle S_j, \psi \rangle = \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \langle \check{S}_j * \psi_i, \check{\phi}_\epsilon \rangle.$$

We claim that one can interchange the limits on the right-hand side of (7). First we fix  $j$  and discuss the interchange of the limits in  $i$  and  $\epsilon$ . By Lemma 1 the set  $\{S_j * \psi_i : i = 1, 2, 3, \dots\}$  is bounded in  $O_c$ , hence bounded in  $E$ . Since  $(\phi_\epsilon)$  converges to  $\delta$  in  $E'$  it follows that  $\langle S_j * \psi_i, \phi_\epsilon \rangle$  converges to  $\langle S_j * \psi_i, \delta \rangle$  uniformly in  $i$ , and since  $S_j * \psi_i \rightarrow S_j * \psi$  in  $O_c$  it follows that (7) becomes

$$(8) \quad \lim_{j \rightarrow \infty} \langle S_j, \psi \rangle = \lim_{j \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \lim_{i \rightarrow \infty} \langle \check{S}_j * \psi_i, \check{\phi}_\epsilon \rangle.$$

Next we consider the interchange of the limits in  $j$  and  $\varepsilon$  in (8). From Lemma 3 it follows that the set  $\{\check{S}_j * \psi : j = 1, 2, 3, \dots\}$  is bounded in  $O_c$ , hence bounded in  $E$  and (8) becomes

$$(9) \quad \lim_{j \rightarrow \infty} \langle S_j, \psi \rangle = \lim_{\varepsilon \rightarrow 0} \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \langle \check{S}_j * \psi_i, \check{\phi}_\varepsilon \rangle.$$

Finally we consider the interchange of the limits of (9) in  $i$  and  $j$ . Since  $S_j \rightarrow 0$  in  $(O'_c, \tau'_b)$  it follows that  $\lim_{j \rightarrow \infty} \langle \check{S}_j * \psi_i, \check{\phi}_\varepsilon \rangle = 0$  uniformly in  $i$  (since  $\{\psi_i : i = 1, 2, \dots\}$  is a bounded subset of  $O_c$  which is continuously embedded in  $K'_M$ , hence it is a bounded subset of  $K'_M$ , and for fixed  $\phi_\varepsilon$  the sequence  $(\check{S}_j * \psi_i)$  converges to 0 in  $K'_M$  uniformly in  $i$ ). Thus one has

$$\lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \langle \check{S}_j * \psi_i, \phi_\varepsilon \rangle = \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \langle \check{S}_j * \psi_i, \phi_\varepsilon \rangle = 0.$$

Hence

$$\lim_{j \rightarrow \infty} \langle S_j, \psi \rangle = \lim_{\varepsilon \rightarrow 0} \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \langle \check{S}_j * \psi_i, \check{\phi}_\varepsilon \rangle = 0,$$

i.e.  $(S_j)$  converges to 0 in  $(O'_c, \text{sdt})$ . This completes the proof of the theorem.

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