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ON THE TOPOLOGY OF THE SPACE OF CONVOLUTION OPERATORS IN K'_M

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ABSTRACT. In this paper we show that on the space $O'_c(K'_M : K'_M)$ of convolution operators on K'_M, the topology τ_b of uniform convergence on bounded subsets of K_M is equal to the strong dual topology.

INTRODUCTION

In previous work (see [1, 2]) we redefined the space K'_M of rapidly increasing distributions and the space $O'_c(K'_M : K'_M)$ of its convolution operators. The space $O'_c(K'_M : K'_M)$ was provided with several topologies, the topology τ_b of uniform convergence on bounded subsets of K_M, the topology of τ'_b of uniform convergence on bounded subsets K'_M, the projective limit topology τ_p where $O'_c(K'_M : K'_M)$ was considered as the projective limit of the spaces $e^{-M(kx)}S'$, and the strong dual topology where $O'_c(K'_M : K'_M)$ was considered as the dual of $O_c(K_M : K_M)$ (see definitions below). It was shown in [2] that τ_b and τ'_b are equal, and τ_p is equal to the strong dual topology. The question whether τ_b and the strong dual topology are equal or not was left unanswered in [2]. Our main result in this paper is the following.

Theorem. *On the space $O'_c(K'_M : K'_M)$ the topology τ_b is equal to the strong dual topology.*

To establish this result we need the following

Lemma. *The convolution map from $(O'_c, \tau_b) \times O_c$ into O_c is separately continuous.*

To avoid lengthy proofs we will present these results in several steps.

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Notations and preliminary results

By N^n, R^n we denote the sets of n-tuples of nonnegative integers and real numbers, respectively. For $\alpha = (\alpha_1, \dots, \alpha_n)$ in N^n we denote by $|\alpha|$ the sum $\alpha_1 + \cdots + \alpha_n$. By D and D' we denote the Schwartz spaces of test functions and distributions, by S we denote the space of infinitely differentiable functions rapidly decreasing at infinity and its strong dual S' is the space of tempered distribution. For any distribution T we denote by $\check{T}$ its symmetry with respect to the origin and by $\tau_h T$, $h \in R^n$, the translation of T by h. For $\alpha \in N^n$ we denote by D the differential operator $D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$; where

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}; \qquad j = 1, 2, \dots, n.$$

Let E be a locally convex topological vector space and E' its strong dual, for a bounded subset B of E we denote by B° the polar of B, which is the set of all T in E' such that $|\langle T, \phi\rangle| < 1$ for all ϕ in B. We will use the results of our previous work [1] and [2].

Let $\mu(\xi)$; $0 \le \xi \le \infty$ be a nonnegative increasing function with $\mu(0) = 0$, $\mu(\infty) = \infty$. For any x, $0 \le x < \infty$ define the function $M(x) = \int_0^x \mu(\xi)d\xi$. It follows that M is positive, increasing, and convex. For negative x we define M by symmetry, i.e. $M(x) = M(-x)$. For $x = (x_1, x_2, \dots, x_n)$ we define $M(x) = M(x_1) + M(x_2) + \cdots + M(x_n)$. The function M satisfies the inequality $M(x+y) \le M(2x) + M(2y)$ for all x, y in R^n. We will write w^k and w^{-k} for the function $\exp[M(kx)]$ and $\exp[-M(kx)]$, respectively; $k = 0, 1, 2, \dots$.

The space K_M consists of all C^∞-functions ϕ such that

$$\nu_k(\phi) = \sup_{\substack{x \in R^n \\ |\alpha| \le k}} w^k \left|D^\alpha \phi(x)\right|; \qquad k = 0, 1, 2, \dots.$$

It has been proved in Abdullah [1] that K_M is a Fréchet nuclear space, moreover it is Montel (hence reflexive), bornologic and is a normal space of distributions. By K'_M we denote the strong dual of K_M provided with the topology of uniform convergence on bounded subsets of K_M, K'_M is the space of distributions which grow not faster than $\exp(M(kx))$ for some $k \ge 0$. The elements of K'_M are called distributions of rapid growth. In the case $M(x) = |x|^p/p$; $p > 1$, the spaces K_M, K'_M are the spaces K_p and K'_p of Sampson and Zielezny [4]. It turns out that K'_M is bornologic. For $T \in K'_M$ and $\phi \in K_M$ we define the convolution of T and ϕ by the relation $(T * \phi)(x) = \langle T_y, \phi(x-y)\rangle$.

The following theorem (see [2], Theorem 1) will be used later in the proofs; its proof is similar to the proof of the corresponding result for the special case $M(x) = |x|^p/p$, $p > 1$ and will be omitted (see Theorem 2 of [4]).

Theorem A. *Let S be any element of K'_M, the following statements are equivalent.*

(a) *S is in $O'_c(K'_M : K'_M)$.*

(b) *The distribution* $w^k S$, $k = 0, 1, 2, \ldots$, *are in* S'.

(c) *For any* $k \geq 0$ *there exists a nonnegative integer* m *such that*

$$S = \sum_{|\alpha| \leq m} D^\alpha f_\alpha ,$$

where, for each α, f_α *is a continuous function such that* $w^k f_\alpha \in L^\infty$.

(d) *For any* k, *the set of distributions* $\{w^{-k}(h)\tau_h S \colon h \in R^n\}$ *is bounded in* D'.

(e) $S * \phi$ *is in* K_M *for all* ϕ *in* K_M *and the map* $\phi \to S * \phi$ *from* K_M *into* K_M *is continuous.*

For $S \in O'_c(K'_M \colon K'_M)$ and $T \in K'_M$ we define $S * T$, the convolution of S and T, by $\langle S * T , \phi \rangle = \langle T , \check{S} * \phi \rangle$; $\phi \in K_M$, where $(\check{S} * \phi)(x) = \langle S_y , \check{\phi}(x - y) \rangle$. Let (T_j) be a sequence in K'_M converging to 0; from property (3) of Theorem 5 of [1] it follows that $S * B$ is bounded in K_M for every bounded subset B of K_M. Hence $\langle S * T_j , \phi \rangle = \langle T_j , \check{S} * \phi \rangle \to 0$ uniformly in $\phi \in B$. Since K'_M is bornological it follows that the map $T \to S * T$ from K'_M into K'_M is continuous. The space $O'_c(K'_M \colon K'_M)$ is the space of convolution operators in K'_M.

The space $O'_c(K'_M \colon K'_M)$ will be provided with several topologies. The first topology τ_p is the projective limit topology of the spaces $w^{-k} S'$, a second topology τ_b induced by $L_b(K_M \colon K_M)$, the space of all continuous linear maps from K_M into itself provided with the topology of uniform convergence on bounded subsets of K_M, and a third topology τ'_b is induced by $L_b(K'_M \colon K'_M)$, the space of all continuous linear maps from K'_M into itself provided with the topology of uniform convergence on bounded subsets of K'_M.

We denote by $O_c(K'_M \colon K'_M)$ the union of the spaces $w^k S$, $k = 0, 1, 2, \ldots$, provided with the inductive limit topology. Since S is bornological (being metrizable), and the inductive limit of bornological spaces is bornological, it follows that $O_c(K'_M \colon K'_M)$ is bornological. The space $O_c(K'_M \colon K'_M)$ is the strong dual of $O'_c(K'_M \colon K'_M)$ with the topology τ_p, (see [2], Theorem 3). Moreover, on $O_c(K'_M \colon K'_M)$ the strong dual topology, i.e. the topology of uniform convergence on bounded subsets of $O'_c(K'_M \colon K'_M)$, is equivalent to the inductive limit topology (see [2], Theorem 5).

On $O'_c(K'_M \colon K'_M)$, the topologies τ_b and τ'_b are equivalent (see [1], Theorem 6), and with τ_b the space $O'_c(K'_M \colon K'_M)$ is complete and nuclear. Moreover, $O'_c(K'_M \colon K'_M)$ with τ_b is bornological, the proof is similar to the space $O'_c(K'_1 \colon K'_1)$ of [6]. Another topology on $O'_c(K'_M \colon K'_M)$ is the strong dual topology, which is the topology of uniform convergence on bounded subsets of $O_c(K'_M \colon K'_M)$. We denote by (O'_c, sdt) the space $O'_c(K'_M \colon K'_M)$ provided with the strong dual topology. The strong dual topology of $O'_c(K'_M \colon K'_M)$ is equivalent to the projective limit topology τ_p (see [2], Theorem 4). From now on, we will write O_c, O'_c for $O_c(K'_M : K'_M)$ and $O'_c(K'_M : K'_M)$, respectively.

THE RESULTS

We begin with a preliminary result that will be used in the proofs of the subsequent results.

Theorem 1. *For any T in K'_M and ϕ in K_M the convolution $T * \phi$ is in O_c, moreover the bilinear map Λ from $K'_M \times K_M$ into O_c mapping (T, ϕ) to $T * \phi$ is separately continuous in both variables.*

Proof. Let T be any element in K'_M, by the characterization theorem of elements of K'_M it follows that there exist a multi-index α, a positive integer k, and a bounded continuous function f such that $T = D^\alpha[w^k f(x)]$. For any multi-index β, from the properties of M, one has

$$\begin{aligned} \left|D^\beta (T * \phi)(x)\right| &= \left|(-1)^{|\alpha|} \int w^k(y) f(y) D^{(\alpha+\beta)} \phi(x-y)\, dy\right|; \\ &\le c_1 w^{2k}(x) \int w^{2k}(x-y) \left|D^{(\alpha+\beta)}\phi(x-y)\right| dy; \\ &\le c w^{2k}(x), \end{aligned} \tag{1}$$

where k, c are constants and k is independent of β. Hence $T * \phi \in O_c$. Next, we prove continuity of Λ in the second variable. For fixed T in K'_M let Λ_T be the linear map from K_M into O_c taking ϕ to $T * \phi$. Since K_M is bornological, from Theorem II.8.3 of Schaefer [5] it suffices to show that Λ_T is sequentially continuous. Let (ϕ_j) be a sequence in K_M converging to 0, from inequality (1) it follows that the set $\{T * \phi_j;\ j = 1, 2, \ldots\}$ is contained in $w^{4k}S$, for that fixed k. Moreover, for any polynomial $p(x)$ and any positive integer m one has

$$\begin{aligned} &\sup_{\substack{x\in R^n \\ |\beta|\le m}} |p(x)| \left|D^\beta \left(w^{-4k}(x) \left(T * \phi_j\right)(x)\right)\right| \\ &\quad \le c_2 c_k \sup_{\substack{x\in R^n \\ |\beta|\le m \\ \alpha\le\beta}} |p(x)|\, w^{-4k}(x)\, w^{2k}(x) \int w^{2k}(y) \left|D^{\alpha+\beta}\phi_j(y)\right| dy. \end{aligned} \tag{2}$$

The convergence of (ϕ_j) to 0 in K_M implies that the integral converges to 0 as j goes to infinity. Since $\sup_{x\in R^n} |p(x)| w^{-2k}(x) < \infty$, inequality (2) implies that $(w^{-4k} T * \phi_j)$ converges to 0 in S, hence $(T * \phi_j)$ converges to 0 in $w^{4k}S$. Thus $(T * \phi_j)$ converges to 0 in O_c. This establishes the continuity of Λ_T.

Finally, we show continuity of Λ in the first variable. For fixed $\phi \in K_M$ let Λ_ϕ be the linear map from K'_M into O_c taking T to $T * \phi$; since K'_M is bornological it suffices to show that Λ_ϕ is sequentially continuous. Let (T_j)

be a sequence in K'_M converging to 0, we show $\Lambda_\phi(T_j) = T_j * \phi \to 0$ in O_c. We claim that Λ_ϕ from O'_c into K_M is continuous. For, given U any neighborhood of 0 in K_M, consider the set $V(\{\phi\}, U) = \{S \in O'_c : \phi * S \in U\}$, $V(\{\phi\}, U)$ is a member of 0-neighborhood base in (O'_c, τ_b), and $\Lambda_\phi^{-1}(U) = V(\{\phi\}, U)$, this proves the claim. Let B' be any bounded subset of O'_c, the above assertion implies that $\Lambda_{\check{\phi}}(B') = \{\check{\phi} * S : S \in B'\}$ is bounded in K_M. Hence

$$\left\langle \Lambda_\phi \left(T_j\right), S\right\rangle = \left\langle T_j, \check{\phi} * S\right\rangle \to 0 \text{ uniformly in } S \in B',$$

i.e. $\Lambda_\phi(T_j)$ converges to 0 in O_c. This completes the proof of the theorem.

The next theorem represents one direction of the main result in the paper, before we give the proof of the other half we need a few lemmas.

Theorem 2. *On the space O'_c the topology τ_b is weaker than the strong dual topology.*

Proof. Let

$$V(B, U) = \left\{S \in O'_c : S * \phi \in U \text{ for all } \phi \text{ in } B\right\};$$

where U is a neighborhood of 0 in K_M and B is a bounded subset of K_M, be a member of a 0-neighborhood base of τ_b. Since K_M is reflexive we can assume without loss of generality that $U = (B')^\circ$, the polar of B' a bounded subset of K'_M. Also, we can assume that $B = \check{B}$ and $B' = \check{B}' = \{\check{T} : T \in B'\}$. For every T in B' consider the linear map Λ_T from K_M into O_c taking ϕ to $T * \phi$, set $\Gamma = \{\Lambda_T : T \in B'\}$. From Theorem 1 it follows that for each T in B' and each ϕ in K_M the linear maps Λ_T and Λ_ϕ are continuous, hence for each ϕ in K_M the orbit $\Gamma(\phi) = \Lambda_\phi(B') = \{T * \phi : T \in B'\}$ is bounded in O_c. Hence, the set of all ϕ in K_M such that $\Gamma(\phi) = \{\Lambda_T(\phi) = T * \phi : T \in B'\}$ is bounded in O_c consists of all of K_M. Since K_M is a complete metric space it is of second category. From the Banach–Steinhaus theorem (see [3], Theorem 2.5, p. 43) it follows that Γ is an equicontinuous family. By Theorem 2.4 of [3] there exists a bounded subset B_c of O_c such that $\Lambda_T(B) \subset B_c$ for ech Λ_T in Γ, i.e. $T * B$ is contained in B_c for each T in B'. Hence, $B' * B = \bigcup_{T \in B'} T * B \subset B_c$. Being a subset of a bounded subset of O_c it follows that $B * B'$ is bounded in O_c. To complete the proof of the assertion it suffices to show that $V(B, U) = (B' * B)^\circ$. For this, let $S \in (B' * B)^\circ$, one has $|\langle S * \phi, T\rangle| = |\langle S, \check{\phi} * T\rangle| < 1$, for all ϕ in B and all T in B', i.e. $S \in V(B, U)$. Conversely, if $S \in V(B, U)$ then $S * \phi \in (B')^\circ$ for all ϕ in B, i.e. $|\langle S * \phi, T\rangle| = |\langle S, \check{\phi} * T\rangle| < 1$, for all ϕ in B and T in B'. Hence $S \in (B' * B)^\circ$.

Lemma 1. *Let S be any element of O'_c. The convolution map Λ_S from O_c into itself which maps ψ to $S * \psi$ is continuous.*

Proof. First we show that $S * \psi$ is in O_c whenever ψ is in O_c. Since $O_c = \bigcup_{k=0}^\infty w^k S$ it follows that $\psi = w^k \phi$, for some ϕ in S and nonnegative integer

k. From the characterization theorem of O_c' (Theorem A) it follows that there exists a nonnegative integer $m = m(k)$ such that $S = \sum_{|\alpha|\le m} D^\alpha f_\alpha$; where the f_α's are functions with $w^{2k} f_\alpha$ in L^∞. Thus for any multi-index β one has

$$
\begin{aligned}
\left|D^\beta (S * \psi)(x)\right| = \left|S * \left(D^\beta \psi\right)(x)\right| &= \left|\sum_{|\alpha|\le m} \left\langle D^\alpha f_\alpha(y)\,,\, D^\beta \psi(x-y)\right\rangle\right|; \\
&= \left|\sum_{|\alpha|\le m} (-1)^{|\alpha|} \left\langle f_\alpha(y)\,,\, D^{\alpha+\beta}\psi(x-y)\right\rangle\right|; \\
&= \left|\sum_{|\alpha|\le m} (-1)^{|\alpha|} \int f_\alpha(y)\, D^{\alpha+\beta}\psi(x-y)\, dy\right|; \\
&\le \sum_{|\alpha|\le m} \int |f_\alpha(y)| \left|D^{\alpha+\beta}\psi(x-y)\right| dy; \qquad (3)\\
&\le \sum_{|\alpha|\le m} c_1 \int |f_\alpha(y)|\, w^{2k}(x-y) \left|\phi_{\alpha,\beta}(x-y)\right| dy; \\
&\le c_1 w^{4k}(x) \sum_{|\alpha|\le m} \int w^{4k}(y)\, |f_\alpha(y)| \left|\phi_{\alpha,\beta}(x-y)\right| dy; \\
&\le c_1 c_2 w^{4k}(x) \sum_{|\alpha|\le m} \int \left|\phi_{\alpha,\beta}(x-y)\right| dy; \\
&\le c_1 c_2 c_3 w^{4k}(x) = c w^{4k}(x)\,,
\end{aligned}
$$

where $\phi_{\alpha,\beta}$ is an element of S which depends on ψ, α and β, and c is a constant which depends on α, β and k, and k is independent of β. Hence $S * \psi$ is in O_c.

Next we show the continuity of Λ_S. Since O_c is bornological it suffices to show that it is sequentially continuous. Let (ψ_j) be a sequence converging to 0 in O_c, we show that $(S * \psi_j)$ converges to 0 in O_c. We consider O_c with the inductive limit topology. There exists a fixed $k_1 \ge 0$ such that (ψ_j) converges to 0 in $\omega^{k_1} S$. Once more, using the characterization theorem of the space O_c' and the properties of $M(x)$ one gets

$$
\begin{aligned}
w^{-k_1}(x) \left|\left(S * \psi_j\right)(x)\right| &\le \sum_{|\alpha|\le m} C \sup_{x,y\in R^n} w^{-k_1}(x-y) \left|D^\alpha \psi_j(x-y)\right| \\
&\qquad \times \int w^{-k_1}(y)\, |f_\alpha(y)|\, dy; \qquad (4)\\
&\le C \sup_{z\in R^n} w^{-k_1}(z) \left|D^\alpha \psi_j(z)\right| \int w^{-k_1}(y)\, dy; \\
&\le C \sup_{z\in R^n} w^{-k_1}(z) \left|D^\alpha \psi_j(z)\right|,
\end{aligned}
$$

where C is a constant which is not the same in all the estimates. Inequality (4) implies that $(S * \psi_j)$ is contained in $w^{2k_1}S$ and converges to 0 there (since the right-hand side of (4) converges to 0 in S), hence $(S * \psi_j)$ converges to 0 in O_c.

Lemma 2. *Let S be any element of O'_c. The linear map Λ_S from O'_c into itself taking T to $S * T$ is continuous, where O'_c is provided with the strong dual topology.*

Proof. Let B°, where B is a bounded subset of O_c, be a member of 0-neighborhood base of O'_c. One has

$$\begin{aligned}\Lambda_s^{-1}\left(B^\circ\right) &= \left\{T \in O'_c : S * T \in B^\circ\right\} \\ &= \left\{T \in O'_c : |\langle S * T, \psi\rangle| < 1 \text{ for all } \psi \text{ in } B\right\} \\ &= \left\{T \in O'_c : \left|\left\langle T, \check{S} * \psi\right\rangle\right| < 1 \text{ for all } \psi \text{ in } B\right\} \\ &= \left(\check{S} * B\right)^\circ.\end{aligned}$$

From Lemma 1 it follows that $\check{S} * B$ is bounded in O_c, hence $(\check{S} * B)^\circ$ is a member of 0-neighborhood base of O'_c. Thus Λ_S is continuous.

Lemma 3. *Let ψ be any element of O_c. The linear map Λ_ψ from (O'_c, τ_b) into O_c is continuous.*

Proof. Since (O'_c, τ_b) is bornological space it suffices to show that the map is bounded. Given B_c a bounded subset of (O'_c, τ_b), we show that $\psi * B_c$ is bounded in O_c. Let V be a member of 0-neighborhood base in O_c, without loss of generality we can assume that $V = B_1^\circ$, where B_1 is a bounded subset of (O'_c, sdt), that is, O'_c equipped with the strong dual topology. We find $\lambda_1 > 0$ such that $\lambda_1(\psi * B_c) \subset B_1^\circ$. Since K_M is dense in O_c there exists a sequence (ϕ_i) in K_M which converges to ψ in O_c. Thus for any W-neighborhood of 0 in O_c there exists a positive integer $N(W)$ such that $\phi_i \in W + \psi$ whenever $i > N(W)$. Let $\phi = \phi_{2N(W)}$, and consider the following member of 0-neighborhood base in (O'_c, τ_b)

$$M\left(\{\phi\}, B_1^\circ\right) = \left\{S \in O'_c : S * \phi \in B_1^\circ\right\} = \left\{S \in O'_c : |\langle S * \phi, T\rangle| < 1 \text{ for all } T \in B_1\right\}.$$

Because B_c is bounded in (O'_c, τ_b) there exists $\lambda > 0$ such that $\lambda B_c \subset M(\{\phi\}, B_1^\circ)$, i.e. $\lambda(\phi * B_c) \subset B_1^\circ$.

Let $W = (1/2\lambda)(\check{S} * B_1)^\circ$, from Lemma 2 it follows that $\check{S} * B_1$ is a bounded subset of (O'_c, sdt), hence W is a member of 0-neighborhood base for O_c.

Let $\lambda_1 = \lambda/2$, then for any S in B_c one has

$$\begin{aligned} |\langle \lambda_1 (\psi * S) , T\rangle| &= \lambda_1 \left|\left\langle \psi , \check{S} * T\right\rangle\right| \\ &\leq \lambda_1 \left|\left\langle \phi , \check{S} * T\right\rangle\right| + \lambda_1 \left|\left\langle \psi - \phi , \check{S} * T\right\rangle\right| \\ &= \tfrac{1}{2} |\langle \lambda (\phi * S) , T\rangle| + \tfrac{1}{2}\lambda \left|\left\langle \psi - \phi , \check{S} * T\right\rangle\right| \\ &< \tfrac{1}{2} \cdot 1 + \tfrac{1}{2}\lambda \cdot \tfrac{1}{2\lambda} = \tfrac{3}{4} < 1 , \end{aligned} \tag{5}$$

for all T in B_1. Thus $\lambda_1(\psi * B_c) \subset B_1^\circ$, i.e. $\psi * B_c$ is a bounded subset of O_c.

The next theorem establishes the remaining portion of our main result, in the proof the above lemmas will be used in an essential way.

Theorem 3. *On the space O_c', the topology τ_b' is not weaker than the strong dual topology.*

Proof. This will follow provided we show the continuity of the identity map from (O_c' , τ_b') into (O_c' , sdt). Since (O_c' , τ_b') is bornological it suffices to show that the map is sequentially continuous. Let (S_j) be a sequence converging to 0 in (O_c' , τ_b'), since O_c' with the strong dual topology is the dual of the Montel space O_c, we only need to show that (S_j) converges weakly in (O_c' , sdt). We consider first the case ψ in K_M. Since $S_j * \psi \to 0$ in K_M it follows that $\langle S_j , \psi\rangle = (S_j * \psi)(0) \to 0$, i.e. the assertion is true for ψ in K_M. For general ψ in O_c, there exists a sequence (ψ_j) in K_M such that ψ_j converges to ψ in O_c. Let $(\phi_\varepsilon : \varepsilon > 0)$ be a regularizing sequence in D, i.e. $\phi_\varepsilon \to \delta$ in E' as $\varepsilon \to 0$. Thus one has

$$\lim_{\varepsilon\to 0} \left\langle \check{S}_j * \psi_i , \check{\phi}_\varepsilon \right\rangle = \lim_{\varepsilon\to 0} \left\langle S_j , \phi_\varepsilon * \psi_i \right\rangle = \left\langle S_j , \psi_i \right\rangle ,$$

and by continuity of S_j one can define $\langle S_j , \psi\rangle$ as follows:

$$\begin{aligned} \left\langle S_j , \psi \right\rangle &= \lim_{i\to\infty} \left\langle S_j , \psi_i \right\rangle = \lim_{i\to\infty} \left(S_j * \check{\psi}_i\right)(0) \\ &= \lim_{i\to\infty} \lim_{\varepsilon\to 0} \left\langle \check{S}_j * \psi_i , \check{\phi}_\varepsilon \right\rangle . \end{aligned} \tag{6}$$

Hence

$$\lim_{j\to\infty} \left\langle S_j , \psi \right\rangle = \lim_{j\to\infty} \lim_{i\to\infty} \lim_{\varepsilon\to 0} \left\langle \check{S}_j * \psi_i , \check{\phi}_\varepsilon \right\rangle . \tag{7}$$

We claim that one can interchange the limits on the right-hand side of (7). First we fix j and discuss the interchange of the limits in i and ε. By Lemma 1 the set $\{S_j * \psi_i : i = 1 , 2 , 3 , \ldots\}$ is bounded in O_c, hence bounded in E. Since (ϕ_ε) converges to δ in E' it follows that $\langle S_j * \psi_i , \phi_\varepsilon\rangle$ converges to $\langle S_j * \psi_i , \delta\rangle$ uniformly in i, and since $S_j * \psi_i \to S_j * \psi$ in O_c it follows that (7) becomes

$$\lim_{j\to\infty} \left\langle S_j , \psi \right\rangle = \lim_{j\to\infty} \lim_{\varepsilon\to 0} \lim_{i\to\infty} \left\langle \check{S}_j * \psi_i , \check{\phi}_\varepsilon \right\rangle . \tag{8}$$

Next we consider the interchange of the limits in j and ε in (8). From Lemma 3 it follows that the set $\{\check{S}_j * \psi : j = 1, 2, 3, \ldots\}$ is bounded in O_c, hence bounded in E and (8) becomes

$$\lim_{j\to\infty} \left\langle S_j, \psi \right\rangle = \lim_{\varepsilon\to 0} \lim_{j\to\infty} \lim_{i\to\infty} \left\langle \check{S}_j * \psi_i, \check{\phi}_\varepsilon \right\rangle. \tag{9}$$

Finally we consider the interchange of the limits of (9) in i and j. Since $S_j \to 0$ in (O'_c, τ'_b) it follows that $\lim_{j\to\infty}\langle \check{S}_j * \psi_i, \check{\phi}_\varepsilon\rangle = 0$ uniformly in i (since $\{\psi_i : i = 1, 2, \ldots\}$ is a bounded subset of O_c which is continuously embedded in K'_M, hence it is a bounded subset of K'_M, and for fixed ϕ_ε the sequence $(\check{S}_j * \psi_i)$ converges to 0 in K'_M uniformly in i). Thus one has

$$\lim_{j\to\infty} \lim_{i\to\infty} \left\langle \check{S}_j * \psi_i, \phi_\varepsilon \right\rangle = \lim_{i\to\infty} \lim_{j\to\infty} \left\langle \check{S}_j * \psi_i, \phi_\varepsilon \right\rangle = 0.$$

Hence

$$\lim_{j\to\infty} \left\langle S_j, \psi \right\rangle = \lim_{\varepsilon\to 0} \lim_{i\to\infty} \lim_{j\to\infty} \left\langle \check{S}_j * \psi_i, \check{\phi}_\varepsilon \right\rangle = 0,$$

i.e. (S_j) converges to 0 in (O'_c, sdt). This completes the proof of the theorem.

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