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DARBOUX BAIRE-.5 FUNCTIONS

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ABSTRACT. Let $I = [0, 1]$, and let D denote the points of continuity of a function $f: I \to R$. A *Darboux* function maps each connected set to a connected set. A function is *Baire*-1 (*Baire*-.5) if preimages of open sets are F_σ-sets (G_δ-sets). We show that if f is a Darboux Baire-.5 function, then the graph of the restriction of f to D is a dense subset of the whole graph of f. It is already known that there is a Darboux Baire-1 function which does not satisfy this conclusion.

A classical result from real analysis states that the set D of points of continuity of an arbitrary function $f: I \to R$ is a G_δ-set. In 1966, Jones and Thomas showed that for any function $f: I \to I$ with a connected G_δ-graph, D is also a dense subset of I [2]. Their argument still works for a Darboux function with a G_δ-graph. It is not always the case that a Darboux Baire-1 or even a bounded approximately continuous function f satisfies the stronger property that each point on the graph of f has a point of $f|D$ plotted nearby [1, Chapter II, Theorems 1, 2.4, and 6.5]. However, we show this property is satisfied by the Darboux Baire-.5 functions, which form a subcollection of the Darboux Baire-1 functions.

For a subset A of B in the plane, we say that A is *bilaterally c-dense in* B if in each open neighborhood of any point $(x, y) \in B$ lie c-many points of A to the left and to the right of (x, y).

Theorem. *Suppose $f: I \to R$ is a Darboux Baire-.5 function, and let D denote the set of points at which f is continuous. Then the graph of $f|D$ is bilaterally c-dense in the graph of f.*

Proof. Since the graph of f is a G_δ-set, it is the intersection of a nested sequence of open subsets $G_1, G_2, \ldots$ of $I \times R$. By [2], D is a dense G_δ-subset of I. We first show that the graph of $f|D$ is dense in the whole graph of f. Assume it is not. It follows that there is an open neighborhood $S = (a_1, b_1) \times (c_1, d_1)$ of a point $(x_0, f(x_0))$ in $I \times R$ such that

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$(f|D) \cap \overline{S} = \varnothing$. According to the hypothesis, $f^{-1}(c_1, d_1)$ is a G_δ-set. Since $A = (a_1, b_1) \cap f^{-1}(c_1, d_1)$ is a G_δ-set, it is topologically complete [3]. Let $l_x = \{x\} \times R$. For all rational numbers r in (c_1, d_1), define $h(r, c_1) = \{x \in A: \overline{f} \cap l_x$ meets both $I \times \{r\}$ and $I \times \{c_1\}\}$ and $h(r, d_1) = \{x \in A: \overline{f} \cap l_x$ meets both $I \times \{r\}$ and $I \times \{d_1\}\}$. Each of the sets $h(r, c_1)$ and $h(r, d_1)$ is closed in A, and each x in A belongs, for some value of r, to either $h(r, c_1)$ or $h(r, d_1)$. According to the Baire Category Theorem, there is a rational number r_0 for which either $h(r_0, c_1)$ or $h(r_0, d_1)$—say $h(r_0, d_1)$—is somewhere dense in A. Then there is a subinterval (a_2, b_2) of (a_1, b_1) such that $h(r_0, d_1)$ contains the nonempty set $B = A \cap (a_2, b_2)$.

For all rational numbers $r < s$ in $[r_0, d_1]$, define $H(n, r, s) = \{x \in B$: some component of $l_x - G_n$ meets both $I \times \{r\}$ and $I \times \{s\}\}$. As in [2], it can be shown that $H(n, r, s)$ is closed in B, and each point x of B belongs to some $H(n, r, s)$. Then some $H(n_1, r_1, s_1)$ is somewhere dense in B and therefore contains a nonempty set $C = B \cap (a_3, b_3)$, where (a_3, b_3) is a subinterval of (a_2, b_2). Consequently, f misses the set $C \times (r_1, s_1)$. However, for each $x \in C$, $\overline{f} \cap l_x$ meets $I \times \{r_0\}$ and $I \times \{d_1\}$, and so $\overline{f} \cap l_x$ meets $I \times \{r_1\}$ and $I \times \{s_1\}$. This implies every point of $\{x\} \times (r_1, s_1)$ is a limit point of $f|C$ whenever $x \in C$. Then f meets $C \times (r_1, s_1)$, a contradiction. Therefore $f|D$ is dense in f after all.

Since f is a Darboux function, the graph of f is bilaterally dense in itself. It was shown above that the graph of $f|D$ is dense in the graph of f. Moreover, the graph of $f|D$ is c-dense in itself because D is c-dense in itself. It now follows that the graph of $f|D$ is bilaterally c-dense in the graph of f.

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