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ON ALMOST EINSTEIN HOLOMORPHIC VECTOR BUNDLES OVER HERMITIAN SURFACES

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ABSTRACT. We study holomorphic vector bundles (E, h) of rank 2 over a compact Hermitian surface (M, g). Then the notion of a metric with a k-pinched Ricci curvature is introduced and it represents the generalization of the Einstein condition. Some necessary topological conditions for existence of a metric h with k-pinched $(0 \leq k \leq 1)$ Ricci curvature are obtained.

1. INTRODUCTION

Let (E, h) be a complex vector bundle over a 4-dimensional compact Riemannian manifold (M, g). Suppose that E admits a complex connection D whose Ricci curvature satisfies the Einstein condition. We may then ask what relations exist between the Chern numbers of E.

Results of this kind are obtained in [1], [4], and [5] for the case of the tangent bundle, and in [3], [7], and [10] for the general case. In this paper, we generalize these results to the case when the Ricci curvature is "nice," that is, for example, when it is k-pinched. For tangent bundles, we find discussed in [11] a topological obstruction for the existence of metrics with k-pinched Ricci curvature.

The main result in this paper is Theorem 3.2 which generalizes the result of Lübke (see [10]). We established the inequality

$$c_2(E) \geq \left\{ \frac{1}{4} - \frac{(1-k)^2}{16\delta} \right\} c_1^2(E)$$

for a holomorphic vector bundle of rank 2 with k-pinched Ricci curvature. We also study when the equality holds in the above.

In §4 we provide some interesting examples of k-pinched metrics. Let r_1 and r_2 be the eigenvalues of the Ricci endomorphism of (E, h), $\operatorname{rank} E = 2$. To obtain these examples we characterize the conformal classes which contain metric h such that r_1 and r_2 are proportional with some constant factor k.

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2. Preliminaries

In this section we will follow [8, Chapter IV]. Let (E, h) be a holomorphic Hermitian vector bundle of rank λ over an Hermitian manifold (M, g) of complex dimension n. Then (E, h) admits a unique Hermitian connection D and its curvature R is a $(1, 1)$-form with values in the bundle $\mathrm{End}(E)$. If $s = (s_1, \dots, s_\lambda)$ is a local frame field for E, the curvature form $\Omega = (\Omega^i_j)$ with respect to s is given by

$$R(s_j) = \sum \Omega^i_j s_i , \qquad \Omega^i_j = \sum R^i_{j\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta$$

in terms of a local coordinate system $(z^1, \dots, z^n)$ of M. In this paper we assume summation for every pair of repeated indexes. We write

$$h_{ij} = h(s_i, s_j) \qquad \text{and} \qquad g = \sum g_{\alpha\overline{\beta}} dz^\alpha d\overline{z}^\beta .$$

Now we define ρ and $\widehat{\rho}$, the Ricci and $\widehat{\ }$-Ricci curvatures of (E, h) respectively, by

$$\rho^i_j = \sum g^{\alpha\overline{\beta}} R^i_{j\alpha\overline{\beta}} , \qquad \rho_{j\overline{k}} = \sum h_{i\overline{k}} \rho^i_j \tag{2.1}$$

and

$$\widehat{\rho}_{\alpha\overline{\beta}} = \sum h^j_i R^i_{j\alpha\overline{\beta}} , \qquad \widehat{\rho}^{\,\beta}_{\alpha} = \sum \widehat{\rho}_{\alpha\overline{\gamma}} g^{\overline{\gamma}\beta} . \tag{2.2}$$

Suppose now that $s = (s_1, \dots, s_\lambda)$, $e = (e_1, \dots, e_n)$, and $\theta = (\theta^1, \dots, \theta^n)$ are local unitary frame fields for (E, h), TM and T^*M respectively. Then we write

$$\Phi = \sqrt{-1} \sum \theta^\alpha \wedge \overline{\theta}^\alpha$$
$$\|R\|^2 = 4 \sum \left| R^i_{j\alpha\overline{\beta}} \right|^2$$
$$\|\rho\|^2 = 2 \sum \left| \rho^i_j \right|^2 = 2 \sum \left| \rho_{j\overline{i}} \right|^2$$
$$\|\widehat{\rho}\|^2 = 2 \sum \left| \widehat{\rho}_{\alpha\overline{\beta}} \right|^2 = 2 \sum \left| \widehat{\rho}^{\,\beta}_{\alpha} \right|^2$$
$$\tau = 2 \sum \rho^i_i = 2 \sum \widehat{\rho}_{\alpha\overline{\alpha}}$$

for the fundamental form of (M, g), the norms of the tensors R, ρ, $\widehat{\rho}$, and for the scalar curvature τ of (E, h) respectively. Scalar curvatures and $\widehat{\ }$-Ricci tensors of holomorphic vector bundles (E', h') and (E'', h'') are denoted by τ', τ'', $\widehat{\rho}'$ and $\widehat{\rho}''$ respectively. The norms of the tensors $\rho - \frac{\tau}{4} h$ and $\widehat{\rho}' - \widehat{\rho}''$ are defined by

$$\left\| \rho - \frac{\tau}{4} h \right\|^2 = 2 \sum \left| \rho_{j\overline{j}} - \frac{\tau}{4} h_{j\overline{j}} \right|^2$$
$$\|\widehat{\rho}' - \widehat{\rho}''\|^2 = 2 \sum |\widehat{\rho}'_{\alpha\overline{\alpha}} - \widehat{\rho}''_{\alpha\overline{\alpha}}|^2 .$$

From now on we suppose $\lambda = n = 2$. Then let ϱ and $\widehat{\varrho}$ denote the sections of the bundles $\mathrm{End}(E)$ and $\mathrm{End}(TM)$ defined by

$$h(\varrho(s), t) = \rho(s, t) \qquad \text{and} \qquad g(\widehat{\varrho}(e), f) = \widehat{\rho}(e, f)$$

for $s, t \in E_p$ and $e, f \in TM_p, p \in M$. The endomorphisms ϱ and $\widehat{\varrho}$ are symmetric, so their corresponding eigenvalues r_1, r_2 and $\widehat{r}_1, \widehat{r}_2$ are real. We will use a local unitary frame field $s = (s_1, s_2)$ determined by the eigenvectors of ϱ corresponding to r_1 and r_2. Also, a local unitary frame field $e = (e_1, e_2)$ is determined by the eigenvectors of $\widehat{\varrho}$.

Let $r = \max\{|r_1|, |r_2|\}$. Then, for $0 \le k \le 1$, we say that Ricci curvature ρ of (E, h) is *k-pinched* if

$$krh \le \rho \le rh \qquad \text{or} \qquad -rh \le \rho \le -krh \tag{2.3}$$

holds on M. Then, clearly, $r_1 r_2 \ge 0$ on M. When $k = 1$, (2.3) represents the weak Einstein condition as defined by Kobayashi.

The Gauss curvature $\widehat{\tau}$ of (E, h) can be defined by

$$\widehat{\tau} = \det \widehat{\varrho} = \widehat{r}_1 \cdot \widehat{r}_2.$$

The Gauss curvature $\widehat{\tau}$ is *δ-bounded from below* if

$$\widehat{\tau} \ge \delta r^2 \tag{2.4}$$

on M. The class of holomorphic vector bundles which satisfy Conditions (2.3) and (2.4) is denoted by $\mathscr{E}_{k,\delta}$.

We now prove the key lemma of the paper.

Lemma 2.1. *Let (E, h) be a holomorphic vector bundle of rank 2 over an Hermitian surface (M, g). Then the following inequality holds*

$$\|R\|^2 \ge \|\widehat{\rho}\|^2 + \|\rho - \tfrac{\tau}{4}h\|^2 . \tag{2.5}$$

Proof. The definition of a norm of the tensor R implies

$$\|R\|^2 \ge \|\widehat{\rho}\|^2 + 2(|R_{1\bar{1}1\bar{1}} - R_{1\bar{1}2\bar{2}}|^2 + |R_{2\bar{2}1\bar{1}} - R_{2\bar{2}2\bar{2}}|^2) .$$

Because of

$$2(|R_{1\bar{1}1\bar{1}} - R_{1\bar{1}2\bar{2}}|^2 + |R_{2\bar{2}1\bar{1}} - R_{2\bar{2}2\bar{2}}|^2) \ge |\rho_{11} - \rho_{22}|^2$$

and

$$\|\rho - \tfrac{\tau}{4}h\|^2 = |\rho_{11} - \rho_{22}|^2$$

we have (2.5). □

When (E, h) satisfies the weak Einstein condition, i.e. ρ is a 1-pinched, the equality case was studied in [8]. It was shown that the equality holds if and only if (E, h) is projectively flat, i.e. $R = \frac{1}{2}\widehat{\rho} \otimes h$. So it is natural to study when the equality holds in the general case.

Lemma 2.2. *Let (E, h) be a holomorphic vector bundle of rank 2 over an Hermitian surface (M, g) such that the Ricci curvature tensor ρ is parallel and $r_1 \ne r_2$ on M. Then the equality*

$$\|R\|^2 = \|\widehat{\rho}\|^2 + \|\rho - \tfrac{\tau}{4}h\|^2 \tag{2.6}$$

holds if and only if E *splits*

$$E = E' \oplus E''$$

for some holomorphic orthogonal line bundles (E', h'), (E'', h'') *in such a way that*

$$\widehat{\rho}' - \widehat{\rho}'' = \tfrac{1}{4}(\tau' - \tau'')g \; . \tag{2.7}$$

Remark. If $r_1 = r_2$ on M the same conditions imply the equality (2.6).

Proof. Suppose that the condition (2.7) is satisfied. Then for $E = E' \oplus E''$ we have

$$\|R\|^2 = 2(\|\widehat{\rho}'\|^2 + \|\widehat{\rho}''\|^2) \, , \qquad \widehat{\rho} = \widehat{\rho}' + \widehat{\rho}'' \, ,$$

$$\tau' = 2\rho_{11} \qquad \text{and} \qquad \tau'' = 2\rho_{22} \, .$$

Thus,

$$\|R\|^2 - \|\widehat{\rho}\|^2 - \|\rho - \tfrac{\tau}{4}h\|^2 = \|\widehat{\rho}' - \widehat{\rho}''\|^2 - \tfrac{1}{4}|\tau' - \tau''|^2 = 0 \, .$$

Assume now that the equality holds and $r_1 \neq r_2$ on M. Then the subbundles E' and E'' of E can be defined by

$$E'_p = \{t \in E_p | \varrho(t) = r_1 t\}$$

$$E''_p = \{t \in E_p | \varrho(t) = r_2 t\}$$

for $p \in M$. Since the Ricci curvature tensor ρ is parallel, (E', h') and (E'', h'') are holomorphic line bundles and $E = E' \oplus E''$ is a global decomposition of E by Theorem 3.21 in [8]. The Hermitian products h' and h'' are determined as restrictions of h. Then $R = R' + R''$ and by the first part of the proof we get

$$\|(\widehat{\rho}' - \widehat{\rho}'') - \tfrac{1}{4}(\tau' - \tau'')g\|^2 = 0$$

i.e., the Einstein-type condition (2.7) is satisfied. □

For the Chern numbers $c_1^2(E)$ and $c_2(E)$ of E the following formulas are obtained in [8, p. 113]

$$c_1^2(E) = \frac{1}{8\pi^2} \int_M (\tau^2 - 2\|\widehat{\rho}\|^2)\Phi^2 \tag{2.8}$$

and

$$c_2(E) = \frac{1}{16\pi^2} \int_M (\tau^2 - 2\|\rho\|^2 - 2\|\widehat{\rho}\|^2 + \|R\|^2)\Phi^2 \, . \tag{2.9}$$

We will now mention some consequences of the previous results.

Corollary 2.3. *Let* (E, h) *be a holomorphic vector bundle of rank* 2 *over a compact Hermitian surface* (M, g). *If the Gauss curvature of* E *is nonnegative, then*

$$c_1^2(E) \geq 0 \, .$$

Proof. The Chern number $c_1^2(E)$ can be expressed in terms of the Gauss curvature. Since $\tau = 2(\widehat{r}_1 + \widehat{r}_2)$ and $\|\widehat{\rho}\|^2 = 2(\widehat{r}_1^2 + \widehat{r}_2^2)$, we have $\tau^2 - 2\|\widehat{\rho}\|^2 = 8\widehat{r}_1 \cdot \widehat{r}_2$. The Gauss curvature $\widehat{\tau}$ is defined by $\widehat{\tau} = \widehat{r}_1 \cdot \widehat{r}_2$. Hence, $\tau^2 - 2\|\widehat{\rho}\|^2 = 8\widehat{\tau}$ and (2.8) implies

$$c_1^2(E) = \frac{1}{\pi^2}\int_M \widehat{\tau}\Phi^2 .$$

Now the result follows immediately. □

Remark. This result is already known (see [7, Theorem 4.1]) because $\widehat{\tau}$ is nonnegative if and only if $\widehat{\ }$-Ricci curvature is nonnegative or nonpositive.

Corollary 2.4. *Let (E, h) be a holomorphic line bundle over a compact Hermitian surface M and let E^* be the dual bundle of E. Then the holomorphic bundle $E \oplus E^*$ does not admit metrics whose Gauss curvature is positive (or negative) on M.* □

3. Chern numbers $c_1^2(E)$ and $c_2(E)$

Let $e(E)$ denote the Euler characteristic of complex vector bundle E.

Theorem 3.1. *Let (E, h) be a holomorphic vector bundle of rank 2 over a compact Hermitian surface (M, g). If the Ricci tensor is k-pinched and the Gauss curvature is $\frac{1}{4}(1-k)^2$-bounded from below we have*

$$e(E) = c_2(E) \geq 0 .$$

Here, $e(E)$ is the Euler characteristic of E. If the Ricci curvature ρ is parallel and $k < 1$, the equality holds if and only if (E, h) admits a holomorphic orthogonal decomposition $(E, h) = (E', h') \oplus (E'', h'')$ with $\widehat{\rho}' - \widehat{\rho}'' = \frac{1}{4}(1-k)rg$ and $\widehat{\tau} = \frac{1}{4}(1-k)^2r^2$. For $k = 1$ the same conditions imply the equality.

Proof. Using the formulas (2.8) and (2.9) and Lemma 2.1. we have

$$\begin{aligned} e(E) = c_2(E) &= \frac{1}{16\pi^2}\int_M (\tau^2 - 2\|\rho\|^2 - 2\|\widehat{\rho}\|^2 + \|R\|^2)\Phi^2 \\ &\geq \frac{1}{16\pi^2}\int_M \{4\widehat{\tau} - (\rho_{11} - \rho_{22})^2\}\Phi^2 . \end{aligned}$$

Since ρ is k-pinched and $\widehat{\tau}$ is $\frac{1}{4}(1-k)^2$-bounded we get $e(E) \geq 0$.

Assume that the equality holds. Then the equality holds in (2.5) and $\widehat{\tau} = \frac{1}{4}(1-k)^2r^2$ and $|\rho_{11} - \rho_{22}|^2 = (1-k)^2r^2$. Thus $\widehat{\tau}' - \widehat{\tau}'' = (1-k)r$. Then by Lemma 2.2. we complete proof of Theorem 3.1. □

For an Einstein-Hermitian vector bundle (E, h), Lübke [10] (cf. also [8]) established the following inequality

$$c_2(E) \geq \tfrac{1}{4}c_1^2(E) . \tag{3.1}$$

The equality holds in (3.1) if and only if (E, h) is a projectively flat vector bundle.

In our next theorem, we generalize the inequality (3.1) for bundles in the class $\mathscr{E}_{k,\delta}$.

Theorem 3.2. *Let (E, h) be an Hermitian vector bundle of rank 2 over a compact Hermitian surface (M, g) which belongs to $\mathscr{E}_{k,\delta}$, $\delta \geq 0$. Then*

$$c_2(E) \geq \left\{\frac{1}{4} - \frac{(1-k)^2}{16\delta}\right\} c_1^2(E) . \tag{3.2}$$

If the Ricci curvature ρ is parallel and $k < 1$, the equality holds if and only if (E, h) admits a holomorphic orthogonal decomposition

$$(E, h) = (E', h') \oplus (E'', h'') \tag{3.3}$$

with $\widehat{\rho}' - \widehat{\rho}'' = \frac{1}{4}(1-k)rg$ and $\widehat{\tau} = \delta r^2$.

Remark. In the Einstein-Hermitian case, i.e. for k=1, the inequality (3.2) is reduced to the inequality (3.1). The equality holds if (E, h) admits decomposition (3.3) with $\widehat{\rho}' = \widehat{\rho}''$ (see [10]).

Proof. Using (2.8), (2.9), and Lemma 2.1 we get

$$16\pi^2(c_2(E) - ac_1^2(E)) \geq \int_M \left\{4(1-4a)\widehat{\tau} - \left\|\rho - \frac{\tau}{4}h\right\|^2\right\} \Phi^2$$

for any real number a. Then $\tau = 2(r_1 + r_2)$ and

$$\|\rho - \tfrac{\tau}{4}h\|^2 = 2(|r_1 - \tfrac{\tau}{4}|^2 + |r_2 - \tfrac{\tau}{4}|^2 = |r_1 - r_2|^2 .$$

When the Ricci curvature ρ is k-pinched, we easily see that (3.2) implies

$$|r_1 - r_2| \leq (1-k)r .$$

Thus, for $(E, h) \in \mathscr{E}_{k,\delta}$ we have

$$\widehat{\tau} \geq \delta r^2 \qquad \text{and} \qquad \|\rho - \tfrac{\tau}{4}h\|^2 \leq (1-k)^2 r^2 . \tag{3.5}$$

Moreover, (3.4) and (3.5) imply

$$4(1-4a)\widehat{\tau} - \|\rho - \tfrac{\tau}{4}h\|^2 \geq \{4(1-4a)\delta - (1-k)^2\} r^2 .$$

Combining (3.4) and (3.6) we obtain

$$16\pi^2(c_2(E) - ac_1^2(E)) \geq \int_M \{4(1-4a)\delta - (1-k)^2\} r^2 \Phi^2 . \tag{3.7}$$

Notice that $4(1-4a)\delta - (1-k)^2$ vanishes for $a = \frac{1}{4} - (1-k)^2/16\delta$. Thus, substituting $a = \frac{1}{4} - (1-k)^2/16\delta$ in (3.7) we obtain the inequality (3.2).

Now, assume that the equality holds in (3.2). Then the equality holds in (2.5) and

$$\widehat{\tau} = \delta r^2 \qquad \text{and} \qquad |\rho_{11} - \rho_{22}|^2 = (1-k)^2 r^2 .$$

Thus $\widehat{\tau}' - \widehat{\tau}'' = (1-k)r$. Then by Lemma 2.2, we complete proof of Theorem 3.2. □

Remark. The inequalities (2.5) and (3.2) also hold for formally holomorphic vector bundles.

4. Conformal change of bundle metric

Let (E, h) be a holomorphic Hermitian vector bundle of rank 2 over a compact Kähler manifold (M, g) of complex dimension n. We choose k to be the constant such that

$$\int_M r_1 \Phi^n = k \int_M r_2 \Phi^n . \tag{4.1}$$

Theorem 4.1. *Let (E, h) be an Hermitian vector bundle of rank 2 over a compact Kähler manifold (M, g) and let k be the constant defined by (4.1). Then, if $k \neq 1$, there is a conformally equivalent Hermitian structure $h' = ah$, a is a real positive function on M, such that*

$$r_1' = k r_2' \tag{4.2}$$

on M. This metric is unique up to homothety.

Proof. Let a be a real, positive function on M and let $h' = ah$ be a new Hermitian structure with the Ricci curvature tensor $\rho'_{j\bar{k}}$ and the corresponding endomorphism ϱ'. Then

$$\rho'_{j\bar{k}} = a\rho_{j\bar{k}} - (\triangle \log a) a h_{j\bar{k}} ; \tag{4.3}$$

where $\triangle = g^{\alpha\bar{\beta}} \partial_\alpha \partial_{\bar{\beta}}$. Because of (4.3) we have

$$r_1' = r_1 - \triangle \log a \qquad \text{and} \qquad r_2' = r_2 - \triangle \log a ,$$

where r_1, r_2 and r_1', r_2' are the eigenvalues of ϱ and ϱ' respectively. Moreover, $r_1 - r_2$ is a conformal invariant.

Then, the differential equation (4.2) can be written as

$$r_1 - k r_2 = (1 - k)(\triangle \log a)$$

or

$$\triangle \log a = f \tag{4.4}$$

where $f = (r_1 - k r_2)/(1 - k)$.

Now, we will establish the existence of a globally defined solution of (4.4). Since $\triangle$ is a self-adjoint elliptic operator of order 2, we can apply Hodge's decomposition theorem (see [6], p. 42) which says:

$$C^\infty(M) = \mathrm{Ker}(\triangle) \oplus \triangle(C^\infty(M)) . \tag{4.5}$$

So, there are the harmonic function f_0 and the smooth function f_1 on M such that

$$f = f_0 + \triangle f_1 . \tag{4.6}$$

The decomposition (4.5) is orthogonal, i.e., $\mathrm{Ker}(\triangle) \perp \triangle(C^{\infty}(M))$, hence we have

$$\int_M \triangle f_1 \Phi^n = 0$$

because constant functions lie in $\mathrm{Ker}(\triangle)$. From (4.1) directly follows $\int f\Phi^n = 0$. Now, using (4.6) we show

$$\int_M f_0 \Phi^n = 0. \tag{4.7}$$

Then, because M is compact, we can apply Hopf lemma to see that f_0 is constant on M. So, (4.7) implies $f_0 \equiv 0$ on M. Then, $a = \exp(f_1)$ is the global solution of the equation (4.4).

It is known that the kernel of the Laplacian $\triangle$ on a compact manifold consists of constant functions. Hence, this implies the uniqueness property and the proof is completed. □

Remark. From the given proof follows that $r_1 - r_2$ is a conformal invariant.

Remark. For $k = 1$, Theorem 4.1 holds if and only if metric h is an Einstein metric.

Corollary 4.2. *Under the same assumptions as in Theorem* 4.1, *if* $\int r_1\Phi^n \cdot \int r_2\Phi^n \geq 0$ *and* $|\int r_1\Phi^n| < |\int r_2\Phi^n|$, (E, h) *has k-pinched Ricci curvature with a constant* k *defined by the formula* (4.1). □

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