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SUMMANDS OF PERMUTATION LATTICES FOR FINITE GROUPS

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ABSTRACT. Let G be a finite group. An effective criterion is given for a $\mathbf{Z}G$-lattice to be a direct summand of a permutation lattice.

Let G be a finite group. A $\mathbf{Z}G$-lattice is by definition a $\mathbf{Z}G$-module which is free and finitely generated as a $\mathbf{Z}$-module. Such a module is called a permutation lattice if it has a $\mathbf{Z}$-basis which is permuted by G, and it is called *invertible* if it is a direct summand, as a $\mathbf{Z}G$-module, of a permutation lattice. These modules have been studied in recent years in connection with questions of rationality of field extensions and function fields of algebraic tori. (An excellent survey is Swan [4]; invertible modules are also called permutation projective, for instance in [2].) Our aim is to give the following criterion for a $\mathbf{Z}G$-lattice to be invertible.

Theorem. *A $\mathbf{Z}G$-lattice M is invertible if and only if it satisfies* I_p *for all primes p and also* II, *where*

- I_p *For a Sylow p-subgroup P of G the restriction $(M/pM)_P$ of M/pM to P is a permutation module for $\mathbf{F}_pP$.*
- II *For a Sylow 2-subgroup P of G the dimensions $\dim F_2(M/2M)^P$ and $\dim_{\mathbf{Q}}(\mathbf{Q}\otimes M)^P$ of P-fixed points are equal.*

We will prove this by first reducing to the case of a p-group, then replacing $\mathbf{Z}$ by the p-adic integers $\mathbf{Z}_p$; for odd primes, the result will follow by using Lemma 3 below. For $p = 2$, Lemma 3 does not hold, and to complete the proof of the theorem we use a result of Weiss [5, Theorem 3]. In an earlier version of this work, we had used Weiss' theorem for all primes; we thank the referee who indicated to us that it could be avoided for odd primes. For results related to Lemmas 1 and 2 see Dress [2].

Lemma 1. *Let M be a $\mathbf{Z}G$-lattice. Then M is invertible if and only if M_P is an invertible $\mathbf{Z}P$-lattice for each Sylow p-subgroup P of G for all primes p.*

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Proof. Let $\mathscr{P}$ be a set containing one Sylow subgroup for each prime dividing the order of G. For each $P \in \mathscr{P}$ there is a $\mathbf{Z}G$-epimorphism ϕ_P: $\mathrm{ind}_P^G(M_P) \to M$ such that $\phi_P(g \otimes m) = gm$, for $g \in G$ and $m \in M$. Let

$$\phi = \bigoplus_{P \in \mathscr{P}} \phi_P \colon \bigoplus_{P \in \mathscr{P}} \mathrm{ind}_P^G(M_P) \to M$$

be the sum of the maps ϕ_P. Since the indices $|G\colon P|$, $P \in \mathscr{P}$, have greatest common divisor 1, we can find integers a_P, $P \in \mathscr{P}$, with $\sum_{P \in \mathscr{P}} a_P |G\colon P| = 1$. Then the epimorphism ϕ is split by the $\mathbf{Z}G$-homomorphism ψ defined by

$$\psi(m) = \sum_{P \in \mathscr{P}} a_P \sum_{g \in G/P} g \otimes g^{-1} m, \qquad m \in M.$$

Therefore, M is isomorphic to a direct summand of $\bigoplus_{P \in \mathscr{P}} \mathrm{ind}_P^G(M_P)$. The lemma now follows easily.

Lemma 2. *Let P be a p-group for a prime p, and let M be a $\mathbf{Z}P$-lattice. Then M is invertible if and only if $\mathbf{Z}_p \otimes M$ is a permutation lattice for $\mathbf{Z}_p P$.*

Proof. Suppose that M is invertible; then $\mathbf{Z}_p \otimes M$ is isomorphic to a direct summand of a permutation $\mathbf{Z}_p P$-lattice. A transitive permutation $\mathbf{Z}_p P$-lattice is indecomposable (which follows, for instance, from Green's indecomposability theorem [1, 19.22]), so $\mathbf{Z}_p \otimes M$ is a permutation lattice, by Krull–Schmidt.

Conversely, suppose that $\mathbf{Z}_p \otimes M$ is a permutation module. We may find a permutation $\mathbf{Z}P$-lattice L with $\mathbf{Z}_p \otimes L \cong \mathbf{Z}_p \otimes M$. Since P is p-group, then L and M are in the same genus, in the sense that their completions over all primes are isomorphic, by [1, 31.2(ii) and 27.1]. By Roiter's Lemma [1, 31.6] there is an exact $\mathbf{Z}P$-sequence

$$0 \to L \to M \to T \to 0,$$

where T is finite of order prime to p. Let F be a free $\mathbf{Z}P$-module mapping onto T, and consider the exact sequence

$$0 \to K \to F \to T \to 0,$$

where K is the kernel of the map from F to T. Then by Roiter's version of Schanuel's Lemma [1, 31.8] we have $M \oplus K \cong L \oplus F$ so M is isomorphic to a direct summand of the permutation lattice $L \oplus F$. This completes the proof.

Lemma 3. *Let P be a p-group, for a prime $p > 2$, and let M be a $\mathbf{Z}_p P$-lattice. Then M is a permutation lattice if and only if M/pM is a permutation module for $\mathbf{F}_p P$.*

Proof. Suppose that M/pM is a permutation module. We claim:

$$H^1(Q, M) = 0 \text{ for all subgroups } Q \text{ of } P.$$

We prove this by induction on $|P|$. If P has order p, then there are only three isomorphism types of indecomposable $\mathbf{Z}_p P$-lattices M (see [1, p. 729]): the trivial module, the free module $\mathbf{Z}_p P$, and the augmentation ideal of $\mathbf{Z}_p P$.

For the first two types $H^1(P, M) = 0$, but since the third is not a permutation module mod p because $p > 2$, it is not considered. Thus the claim holds in this case. For $|P| > p$, only $H^1(P, M) = 0$ must be shown, because the induction hypothesis handles proper subgroups of P. Let Q be a central subgroup of P of order p. The exact sequence $0 \to M \xrightarrow{p} M \to M/pM \to 0$ gives rise to

$$(*) \qquad 0 \to M^Q \xrightarrow{p} M^Q \to (M/pM)^Q \to H^1(Q, M).$$

Since $H^1(Q, M) = 0$ by induction, then $(M/pM)^Q \cong M^Q/p(M^Q)$. Since M/pM is a permutation module for P, then $(M/pM)^Q$ is a permutation module for P/Q. (It has as $\mathbf{F}_p$-basis the Q-orbit sums of a P-permutation $\mathbf{F}_p$-basis of M/pM.) By induction the claim applied to the $\mathbf{Z}_p(P/Q)$-lattice M^Q yields $H^1(P/Q, M^Q) = 0$. Then the inflation–restriction sequence [3, p. 125]

$$0 \to H^1(P/Q, M^Q) \to H^1(P, M) \to H^1(Q, M)^{P/Q}$$

implies that $H^1(P, M) = 0$, and the claim is proved.

Now let $\mathscr{B}$ be a basis of M/pM which is permuted by P. Write $\mathscr{B} = \cup\mathscr{C}$ as a disjoint union of orbits, and take an element y in some orbit $\mathscr{C}$. Let Q be the stabilizer in P of y, and write $P = \cup_{t \in \mathscr{T}} tQ$ as a union of cosets. By the claim and $(*)$, there exists $x \in M^Q$ such that $y = x + pM$. Define $\mathscr{C}' = \{tx : t \in \mathscr{T}\}$, and let $\mathscr{B}' = \cup\mathscr{C}'$. Then $\mathscr{B}'$ is permuted by P, and $\mathscr{B}'$ is a $\mathbf{Z}_p$-basis of M by Nakayama's Lemma, since $\mathscr{B}'$ reduced modulo pM is $\mathscr{B}$. Therefore, M is indeed a permutation $\mathbf{Z}_pP$-lattice. This proves the lemma, since the converse is obvious.

Proof of theorem. By Lemma 1 we may assume that G is a p-group P for some prime p. Then Lemma 2 transfers the problem of deciding if a $\mathbf{Z}P$-lattice M is invertible to deciding if the $\mathbf{Z}_pP$-lattice $L = \mathbf{Z}_p \otimes M$ is a permutation lattice, because $M/pM \cong L/pL$. In case $p = 2$ the condition II can be translated to L because $\mathbf{Q}_2 \otimes L \cong \mathbf{Q}_2 \otimes_{\mathbf{Q}} (\mathbf{Q} \otimes M)$ and $\mathbf{Q}_2 \otimes_{\mathbf{Q}} *$ does not change fixed point dimensions. If $p > 2$ the theorem follows immediately from Lemma 3.

Finally if $p = 2$ and L satisfies I_2 and II then [5, Theorem 3] tells us that $L \cong \bigoplus_{i=1}^n \mathrm{ind}_{P_i}^P((\mathbf{Z}_2)_{\chi_i})$ for certain homomorphisms $\chi_i : P_i \to \{\pm 1\}$, where each P_i is a subgroup of P and $(\mathbf{Z}_2)_{\chi_i}$ denotes the $\mathbf{Z}_2P_i$-lattice with underlying $\mathbf{Z}_2$-module $\mathbf{Z}_2$ and P_i acting via χ_i. Then $L/2L \cong \bigoplus_{i=1}^n \mathrm{ind}_{P_i}^P(\mathbf{F}_2)$, and so $\dim_{F_2}(L/2L)^P = n$, the number of homomorphisms χ_i. On the other hand, $\dim_{\mathbf{Q}_2}(\mathbf{Q}_2 \otimes_{\mathbf{Z}_2} L)^P$ can be computed from the character ξ of L: it is the number of times the trivial character occurs in ξ, which is by Frobenius reciprocity the number of homomorphisms χ_i which are trivial. By hypothesis II, all the χ_i must be trivial, so L is a permutation lattice in this case too. Since the converse is obvious, the proof of the theorem is complete.

Note that the theorem does not hold without hypothesis II: M could be the rank 1 $\mathbf{Z}G$-lattice $\mathbf{Z}$ on which some element of G of even order acts by multiplication by -1. We may define a *signed permutation* lattice for $\mathbf{Z}G$ to be a lattice which has a $\mathbf{Z}$-basis $\{m_i\}$ such that for each g in G, $gm_i = \pm m_j$ for some j. It is not difficult to see that our proof of the theorem can be modified (indeed simplified) to prove the following result.

Theorem. *A $\mathbf{Z}G$-lattice M is a direct summand of a signed permutation lattice if and only if it satisfies* I_p *for all primes* p.

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