PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 110, Number 1, September 1990

DEDEKIND DOMAINS AND GRADED RINGS

FABIEN DECRUYENAERE AND ERIC JESPERS

(Communicated by Maurice Auslander)

ABSTRACT. We prove that a Dedekind domain R, graded by a nontrivial torsionfree abelian group, is either a twisted group ring $k^t[G]$ or a polynomial ring $k[X]$, where k is a field and G is an abelian torsionfree rank one group. It follows that R is a Dedekind domain if and only if R is a principal ideal domain. We also investigate the case when R is graded by an arbitrary nontrivial torsionfree monoid.

We fix some notation and terminology. All rings R are commutative with identity 1, and all semigroups S are torsionfree. In case S is a monoid, we denote by e the identity of S and by $\mathscr{U}(S)$ the group of invertible elements of S. For $s \in S$, denote by $\langle s \rangle$ (respectively $\langle s \rangle^1$) the subsemigroup (respectively submonoid) of S generated by s. For more details on semigroups we refer to [3]. We say that R is S-graded if $R = \bigoplus_{s\in S} R_s$, a direct sum of additive subgroups, such that $R_s R_t \subseteq R_{st}$, for all $s, t \in S$. The set $h(R) = \bigcup_{s\in S} R_s$ is the set of all homogeneous elements. If T is a subset of S, then we put $R_{[T]} = \bigoplus_{t\in T} R_t$. Clearly, if $\mathrm{Supp}(R) = \{s \in S | R_s \neq 0\}$, the support of R, then $R = R_{[\mathrm{Supp}(R)]}$. Obviously $\mathrm{Supp}(R)$ is a monoid if R is a domain. If, moreover, S is a group and $R_s R_t = R_{st}$ for all $s, t \in S$, then R is called strongly S-graded. If I is an ideal of R, we denote by $(I)_h$ the ideal generated by all homogeneous elements of I. If $I = (I)_h$, then I is called a homogeneous ideal of R.

If S is a monoid and R is an S-graded integral domain, then $Q^g(R) = \{rc^{-1} | r \in R,\ 0 \neq c \in R_s,\ s \in S\}$, the graded quotient ring of R; if moreover, S is cancellative, then $Q^g(R)$ is G-graded, where G is the quotient group of S, and its component of degree e is clearly a field. For further details on graded rings we refer to [6].

In recent years there has been a growing interest in divisibility properties of graded rings. For example, in [1, 7] graded rings which are factorial domains are investigated, while in [1, 2] graded rings that are Krull domains are studied. In this paper we investigate graded rings which are Dedekind domains.

Received by the editors April 10, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 13F05, 16A03.
The first author is a research assistant at the National Fund for Scientific Research, Belgium.
The second author is supported in part by NSERC-grant OGP0036631.

We begin with two elementary lemmas.

Lemma 1. *Let G be a torsionfree abelian group, R a G-graded ring and $S = \mathrm{Supp}(R)$. If R is a Dedekind domain, then S is either a group or a torsionfree cancellative monoid with $\mathscr{U}(S) = \{e\}$.*

Proof. Since R is a domain and $1 \in R_e$, S is clearly a submonoid of G. Suppose $S \neq \mathscr{U}(S)$. Since $T = S \backslash \mathscr{U}(S)$ is an ideal of S, it follows that $R_{[T]}$ is an ideal of R; and obviously $R/R_{[T]} \cong R_{\mathscr{U}(S)}$. Hence $R_{[T]}$ is a nonzero prime ideal, and thus a maximal ideal of R. Therefore $R_{\mathscr{U}(S)}$ is a field. Since $\mathscr{U}(S)$ is a totally ordered group, it follows, using well-known techniques, that $\mathscr{U}(S) = \{e\}$. This finishes the proof. □

Lemma 2. *Assume S is a nontrivial torsionfree cancellative monoid and R is an S-graded Dedekind domain. If $S = \mathrm{Supp}(R)$, then the quotient group $\langle S \rangle$ of S has torsionfree rank 1.*

Proof. Let $Q = Q^g(R)$, the graded quotient ring of R. Note that Q is also a Dedekind domain. Since Q is $\langle S \rangle$-graded and because Q_e is a field, it is well known that Q is a twisted group ring $Q_e^t[G]$ of a group G over Q_e. Moreover, $S = \mathrm{Supp}(R)$ yields $G = \langle S \rangle$. Let F be a maximal free subgroup of G, then G/F is a torsion group. Therefore, Q is integral over $Q_e^t[F]$ and since $\dim(Q) = 1$ (note that S is nontrivial) we obtain that $\dim(Q_e^t[F]) = 1$. Now $Q_e^t[F]$ is isomorphic with a Laurent polynomial ring over the field Q_e in $\mathrm{rank}(F)$ variables. It follows that $\mathrm{rank}(F) = 1$. The result follows. □

We consider the two cases mentioned in Lemma 1 separately. First we assume $\mathscr{U}(S) = \{e\}$.

Proposition 3. *Let R be an S-graded ring, where $S = \mathrm{Supp}(R)$ is a nontrivial torsionfree cancellative monoid with $\mathscr{U}(S) = \{e\}$. Then R is a Dedekind domain if and only if $R \cong k[X]$, a polynomial ring over a field k.*

Proof. Let $T = S \backslash \{e\}$. Then T is a nontrivial ideal of S. As in Lemma 1, it follows that $R_{[T]}$ is a maximal ideal of R. Hence $R_e \cong R/R_{[T]}$ is a field.

We first consider the case that $S = \mathbf{N}$, the nonnegative integers. Put $M = R_{[\mathbf{N}_0]} = \bigoplus_{n>0} R_n$, the unique maximal homogeneous ideal of R. Therefore, the ideal generated by R_1 is equal to M^n for some $n \geq 1$. But then $n = 1$, and hence $R_m = R_1 R_{m-1} = R_1^m$, for every $m \geq 1$. Consequently $R = R_0 \oplus \sum_{n \geq 1} R_1^n$. Let $0 \neq r_1 \in R_1$; then Rr_1 is a nonzero homogeneous ideal and thus $Rr_1 = M$. Hence $R_1 = R_0 r_1$ and for each $n \geq 1$, $R_n = R_0 r_1^n$. So $R = \bigoplus_{n \in \mathbf{N}} R_0 r_1^n$, a polynomial ring in r_1 over R_0.

We now consider the general case. Let $s \in S$, $s \neq e$, then $\langle s \rangle^1 \cong \mathbf{N}$. Because of Lemma 2, $G = \langle S \rangle$ has torsionfree rank one. Let $\mathrm{grp}\{s\}$ be the cyclic subgroup of G generated by s, and let $\overline{G} = G/\mathrm{grp}\{s\}$. It follows that R is also a $\overline{G}$-graded ring with identity component $R_{[\langle s \rangle^1]}$. The latter follows from the fact that $\mathscr{U}(S) = \{e\}$. It then follows from [1] that $R_{[\langle s \rangle^1]}$ is a Krull domain.

Now since $\overline{G}$ is a torsion group and because $\mathscr{U}(S) = \{e\}$, we obtain that for every $e \neq t \in S$, there exist $n, m \in \mathbf{N}_0$ such that $t^n = s^m$. Consequently, R is integral over $R_{[\langle s\rangle^1]}$ and, therefore, the latter ring is of dimension 1. Hence $R_{[\langle s\rangle^1]}$ is a Dedekind domain. It follows from the first case that $R_s = R_0 r_s$, $0 \neq r_s \in R_s$. Therefore, $R = R_e^t[S]$, a twisted monoid ring. We now prove that $S \cong \mathbf{N}$; this will finish the proof. Since S is torsionfree cancellative and has no nontrivial invertible elements, there exists a linear order $<$ on S such that all elements of S are positive. We assert that $S\backslash\{e\}$ has a minimum element for this order. For if not, then S has an infinite descending chain

$$s_1 > s_2 > s_3 > \cdots > s_n > \cdots > e.$$

But then one obtains the following infinite strictly ascending chain of ideals in R:

$$\sum_{s\geq s_1} R_s \subset \sum_{s\geq s_2} R_s \subset \cdots \subset \sum_{s\geq s_n} R_s \subset \cdots ;$$

a contradiction. Let s_1 be the minimum element in $S\backslash\{e\}$. So $M = \sum_{s\geq s_1} R_s$ is the unique maximal homogeneous ideal of R and $RR_{s_1} = M$. Consequently, for every $s \in S\backslash\{e\}$, $R_s \subset RR_{s_1}$ and thus $ss_1^{-1} \in S$. If $s_1^n < s < s_1^{n+1}$, $n \geq 1$, then $1 < ss_1^{-n} < s_1$, a contradiction since $ss_1^{-n} \in S$. Therefore,

$$e < s_1 < s_1^2 < \cdots < s_1^n < \cdots$$

is a strictly ascending chain of elements of S which cannot be refined in S. Suppose there exists $t \in S$ such that $t > s_1^n$ for all $n \in \mathbf{N}$. Then by an argument as above, such a minimal element t exists. But then $ts_1^{-1} < t$ and, for every $n \in \mathbf{N}$, $ts_1^{-1} > s_1^n$; a contradiction. Hence $S = \langle s_1\rangle^1 \cong \mathbf{N}$. □

Proposition 4. *Let R be a G-graded ring, where $G = \operatorname{Supp}(R)$ is a nontrivial torsionfree abelian group. If R is a Dedekind domain, then $R \cong k^t[G]$, a twisted group ring over a field k, and G has torsionfree rank one.*

Proof. It follows from Lemma 2 that G is of rank one. Hence to prove the result, it is sufficient to show that R_e is a field, or equivalently that R has no nonzero homogeneous prime ideals. So we assume R_e is not a field and derive a contradiction.

Let P be a nonzero homogeneous prime ideal. Then R/P is a field and a G-graded ring. Therefore R/P is trivially graded, and thus $P = p + \sum_{g\in G\backslash\{e\}} R_g$ where p is a maximal ideal of R_e. Write $P = P(p)$. Conversely, let p be a nonzero prime ideal of R_e. Then Rp is an ideal of R such that $Rp \cap R_e = p$. Let M be a homogeneous ideal of R maximal with respect to $M \cap R_e = p$. One easily verifies that M is a prime ideal of R, and thus by the previous $M = P(p)$. Now fix a nonzero prime ideal p in R_e. Let T be the ring R localized to the multiplicative set $R_e\backslash p$. Then T is also a G-graded Dedekind domain, and by the above T has only one nonzero homogeneous prime ideal, namely $P(p)$ localized to $R_e\backslash p$. We may assume $T = R$.

So let R be a G-graded Dedekind domain with unique nonzero homogeneous prime ideal $P = P(p) = p + \sum_{g\in G\setminus\{e\}} R_g$. It follows that for every $0 \neq x \in P \cap h(R)$ there exists $n(x) \geq 0$ such that $Rx = P^{n(x)}$. Since $P = \sum_{x\in P\cap h(R)} Rx$, we obtain that $P = Rx_g$ for some $x_g \in R_g$, $g \in G$. Assume $g \neq e$. Then $R_g = R_e x_g$ and thus $RR_g = Rx_g = P$. Therefore, for every $h \in G\setminus\{e\}$, $R_h = R_g R_{g^{-1}h}$ and $p = R_g R_{g^{-1}}$. Consequently, $P^k \supseteq p + \sum_{n>0} R_{g^{-n}}$, for all $k \geq 1$; a contradiction as $p \neq 0$. Therefore, $g = e$. In this case it follows that $p = R_e x_e$ and $R_g = R_g x_e$ for $g \neq e$. We obtain $RR_g = RR_g Rx_e$. This yields $R = Rx_e$ and thus $p = R_e$, a contradiction. This finishes the proof. □

Corollary 5. *Let R be a G-graded ring, where G is a torsionfree abelian group with the ascending chain condition on cyclic subgroups; and assume $|\mathrm{Supp}(R)| > 1$. Then, R is a Dedekind domain if and only if $R \cong k[X]$ or $R \cong k[X, X^{-1}]$ for some field k.*

Proof. Since a torsionfree rank one abelian group with the ascending chain condition on cyclic subgroups is free [4], the result follows from Lemmas 1 and 2 and Propositions 3 and 4. □

Remark. Obviously, the condition in Proposition 4 is not sufficient, as, for example, a group algebra over an infinitely generated rank one group is not Noetherian.

Also, G does not need to have the ascending chain condition on cyclic subgroups. The following example is taken from [2]. Let G be an arbitrary torsionfree rank one abelian group and $R = k[X_g | g \in G]$, a polynomial ring over the field k. Obviously R is a unique factorization domain and is G-graded with $\deg(X_{g_1}^{n_1} \cdots X_{g_r}^{n_r}) = g_1^{n_1} \cdots g_r^{n_r}$. Clearly $Q = Q^g(R) = Q_e^t[G]$ is also a unique factorization domain. Moreover, $Q \cong Q_e[\mathbf{Z}]^t[G/\mathbf{Z}]$ is integral over $Q_e[\mathbf{Z}]$; so all nonzero prime ideals of Q are maximal. Therefore, Q is a Dedekind domain.

We obtain the following generalization of a result of Gilmer [5, Theorem 13.8].

Corollary 6. *Let S be a torsionfree cancellative monoid. Assume R is an S-graded ring with $|\mathrm{Supp}(R)| > 1$. The following conditions are equivalent.*

1. *R is a Dedekind domain.*
2. *R is principal ideal domain.*

In these cases, $R \cong k^t[G]$, where G is a torsionfree rank one group, or $R \cong k[X]$ with k a field.

Proof. We only need to observe that, when $R \cong k^t[G] \cong k[\mathbf{Z}]^t[G/\mathbf{Z}]$ (G of rank 1) and R is a Dedekind domain, then R is a principal ideal domain. For this, let P be a prime ideal of R. Then $P = Rr_1 + \cdots + Rr_n$ for some $r_i \in R$. Clearly there exists a finitely generated subgroup H of G such that $r_i \in R_{[H]}$, for all $1 \leq i \leq n$. Since G is torsionfree of rank one, it follows that $H \cong \mathbf{Z}$, $R_{[H]} \cong k[X, X^{-1}]$ and therefore $P \cap R_{[H]} = R_{[H]} r$ for some $r \in R_{[H]}$. Hence $P = Rr$. The result follows. □

We now consider rings graded by noncancellative monoids.

Proposition 7. *Let R be an S-graded ring, where $S = \mathrm{Supp}(R)$ is a nontrivial torsionfree noncancellative monoid. If R is a Dedekind domain, then $|\mathrm{Supp}(R)| = 2$ and $R = k + M$, where k is a field and M is a maximal ideal of R.*

Proof. Since S is torsionfree, $S = \bigcup_{\alpha\in\Gamma} S_\alpha$, the disjoint union of its cancellative Archimedean subsemigroups, with Γ a semilattice. By $\leq$ we denote the partial order relation on Γ, that is $\alpha \leq \beta$ if $\alpha\beta = \alpha$. As S is noncancellative, $|\Gamma| > 1$ and since S is a monoid, Γ has a maximum element, say δ, with $e \in S_\delta$. Moreover, $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$, a semilattice graded ring, where for every $\alpha \in \Gamma$, $R_\alpha = \bigoplus_{s\in S_\alpha} R_s$. Put $R' = \bigoplus_{\alpha\neq\delta} R_\alpha$, then R' is a nonzero prime, and thus a maximal ideal of R. Hence R_δ is a field.

Let $\alpha \neq \delta$ and $P_\alpha = \bigoplus_\beta R_\beta$, where the sum runs over all β that are either incomparable with α or $\beta < \alpha$. Then P_α is an ideal of R and $R/P_\alpha = \bigoplus_{\beta\geq\alpha} R_\beta \neq 0$ is a domain. So P_α is a maximal ideal in R or $P_\alpha = \{0\}$. But in the first case $\bigoplus_{\beta\geq\alpha} R_\beta$ would be a field, which is impossible as each element of $R_\alpha \subseteq R'$ is not invertible. Hence for every $\alpha \neq \delta$, $P_\alpha = \{0\}$, that is $\{\beta \in \Gamma | \beta \leq \alpha$ or β incomparable with $\alpha\} = \varnothing$. Therefore $\Gamma = \{\delta, \alpha\}$, $\alpha \neq \delta$ and $\alpha\delta = \alpha$; $R = R_\delta \oplus R_\alpha$ and $S = S_\delta \cup S_\alpha$. Now if $S_\delta \neq \{e\}$, then S_δ is a nontrivial torsionfree abelian group and R_δ is a field graded by S_δ, which is impossible. So $S = \{e\} \cup S_\alpha$. Now since S is not cancellative there exist $s \in S_\alpha$ and $t, t' \in S$ such that $st = st'$ and $t \neq t'$. But as S_α itself is cancellative we obtain that, say, $t = e$ and thus $t' \in S_\alpha$. This yields that t' is an idempotent, and consequently, $S_\alpha = G$ is a group.

Let 1_G be the identity of G. We claim that $G = \{1_G\}$. For if not, then $R = (R_e + R_{1_G}) \oplus \sum_{g\in G\setminus\{e\}} R_g$ is a Dedekind domain graded by the nontrivial torsionfree group G (the identity component being $R_e + R_{1_G}$). Lemma 4 implies that $R_e + R_{1_G}$ is a field. A contradiction as R_{1_G} is a nonzero ideal of the latter ring. This proves the claim; and therefore $R = R_e + R_{1_G}$. This finishes the proof. □

Note that there are plenty of Dedekind domains of the type $R = k + M$. For example polynomial rings $k[X]$ or power series $k[\![X]\!]$ where k is a field, or $\mathbf{R}[X, Y]/(X^2 + Y^2 - 1)$. In case R is a principal ideal domain we can prove that R is embedded in a formal power series ring and contains a polynomial ring.

Corollary 8. With the notations and assumptions as in Proposition 7. *If, moreover, R is a principal ideal domain and $M = RX$, $X \in R$, then*

$$k[X] \subseteq R = k + M \subseteq k[\![X]\!].$$

Proof. Let $R_e = k$, a field, then $R = k + RX$. It follows that $R = k + kX + kX^2 + \cdots + kX^n + X^n RX$. Thus for every $r \in R$, and $n \geq 0$, there exist

$r_0, r_1, \ldots, r_n \in k$ and $b_n \in RX$ such that $r = r_0 + r_1 X + \cdots + r_n X^n + b_n X^n$. One easily verifies that the r_i's are uniquely determined by r. Hence we obtain a well-defined map

$$\varphi: R \to k[\![X]\!]: r \mapsto r_0 + r_1 X + \cdots + r_n X^n + \cdots .$$

It follows that φ is a ring homomorphism. Moreover, since $\bigcap_{n \in \mathbf{N}} RX^n = \{0\}$, φ is a monomorphism. This proves the result. □

Of course not every principal ideal domain R is of the form $k + M$. Let $R = \mathbf{R}[X]_{(X^2+X+1)}$, that is the localization of $\mathbf{R}[X]$ with respect to the prime ideal generated by $X^2 + X + 1$. Clearly R is a principal ideal domain, and it is easily verified that R is not of the form $k + M$ for some field k and nonzero ideal M.

References

1. D. D. Anderson and D. F. Anderson, *Divisibility properties of graded domains*, Canad. J. Math. **24** (1982), 196–215.
2. D. F. Anderson, *Graded Krull domains*, Comm. Algebra **7** (1979), 79-106.
3. A. J. Clifford and G. B. Preston, *The algebraic theory of semigroups*, vol. I, Math. Surv. Mono. vol. 7, Amer. Math. Soc., Providence, RI, 1961.
4. L. Fuchs,*Infinite abelian groups*, vol. II, Academic Press, New York, 1970.
5. R. Gilmer, *Commutative semigroup rings*, The University of Chicago Press, Chicago and London, 1984.
6. C. Nastasescu and F. Van Oystaeyen, *Graded ring theory*, North-Holland, Amsterdam, 1982.
7. P. Wauters, *Factorial domains and graded rings*, Comm. Algebra (to appear).

Department of Mathematics, K. U. Leuven, 3030 Leuven, Belgium

Department of Mathematics, Memorial University of Newfoundland, St. John's, A1C 5S7 Canada