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QUADRATIC FORMS WITH CUBE-FREE DISCRIMINANT

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ABSTRACT. Formulas are given for the number of genera of **Z**-lattices with rank $n \geq 3$, signature s, and cube-free discriminant Δ. The results are applied to study classification and orthogonal splittings in the indefinite case.

1. INTRODUCTION

Let L be a **Z**-lattice on a regular quadratic space V of finite dimension $n \geq 3$ with signature s and associated symmetric bilinear form $f\colon V \times V \to \mathbf{Q}$. For convenience, assume that $s \geq 0$ and $f(L, L) \subseteqq \mathbf{Z}$. Let $x_1, \dots, x_n$ be a **Z**-basis for L and put $dL = \det f(x_i, x_j)$, the discriminant of the lattice. The lattice is called *even* if $f(x, x) \in 2\mathbf{Z}$ for all $x \in L$, otherwise L is *odd*. We study lattices L with cube-free discriminant; thus $dL = \Delta$ where $\Delta = (-1)^{(n-s)/2}d^2D$, or $\Delta = (-1)^{(n-s)/2}2d^2D$, or $\Delta = (-1)^{(n-s)/2}4d^2D$ with $dD \geq 1$, squarefree and odd. When Δ is odd, even lattices L exist if and only if n is even and $D \equiv (-1)^{s/2} \bmod 4$, and when $\Delta \equiv 2 \bmod 4$, even L exist if and only if n is odd (by Chang [2, Satz 2]).

Denote by $G_e(n, s, \Delta)$ the number of genera of even lattices with rank n, (compatible) signature s, and discriminant Δ, and by $G_0(n, s, \Delta)$ the corresponding number of genera of odd lattices. Let e and f be the number of primes dividing d and D, respectively.

Theorem 1. *Assume* $\Delta = (-1)^{(n-s)/2}d^2D$. *Then for even* $n \geq 4$ *and* $D \equiv (-1)^{s/2} \bmod 4$,

$$G_e(n, s, \Delta) = \begin{cases} 2^{e-1}(2^e + (-1)^{s/4}) & \text{if } D = 1, \\ 2^{2e+f-1} & \text{if } D > 1; \end{cases}$$

and for any $n \geq 3$,

$$G_0(n, s, \Delta) = 2^{2e+f}.$$

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Theorem 2. *Assume* $\Delta = (-1)^{(n-s)/2} 2d^2 D$. *Then for odd* $n \geq 3$,

$$G_e(n, s, \Delta) = \begin{cases} 2^{e-1}(2^e + (-1)^{(s^2-1)/8}) & \text{if } D = 1, \\ 2^{2e+f-1} & \text{if } D > 1; \end{cases}$$

and for any $n \geq 3$,

$$G_0(n, s, \Delta) = 2^{2e+f}.$$

Theorem 3. *Assume* $\Delta = (-1)^{(n-s)/2} 4d^2 D$. *Then for odd* $n \geq 3$,

$$G_e(n, s, \Delta) = 2^{2e+f};$$

for even $n \geq 4$,

$$G_e(n, s, \Delta) = \begin{cases} 2^{e-1}(3 \cdot 2^e + (-1)^{s/4}) & \text{if } D = 1 \text{ and } s \equiv 0 \bmod 4, \\ 2^{2e+f} & \text{if } D \equiv -(-1)^{s/2} \bmod 4, \\ 3 \cdot 2^{2e+f-1} & \text{if } D > 1 \text{ and } D \equiv (-1)^{s/2} \bmod 4; \end{cases}$$

and for any $n \geq 5$, *and for* $n = 4$ *with* $\Delta \equiv 12 \bmod 16$,

$$G_0(n, s, \Delta) = 2^{2e+f+2}.$$

Also $G_0(3, s, \Delta) = 3 \cdot 2^{2e+f}$, *provided* $D > 1$.

In the indefinite case, where $|s| < n$, the class and genus coincide under our assumptions on Δ, and hence these three theorems also determine the number of classes of lattices. Part of Theorem 1 was established in [6]; notation and terminology follow this earlier paper. A knowledge of the classification of lattices is implicit in these three theorems and will now be stated as a separate theorem. For p dividing Δ, let $L_p = L_0(p) \perp L_1(p)$ be a Jordan splitting (see [7, 91C]), where $L_0(p)$ is unimodular and $L_1(p)$ is either p-modular or p^2-modular.

Theorem 4. *Let* L *and* M *be two lattices with the same cube-free discriminant* Δ *on the same quadratic space* V. *Then* L *and* M *are in the same genus (same class when* V *is indefinite) if and only if*

(i) L_p *and* M_p *have the same type for* $p|2d$,

(ii) $dL_0(p) = dM_0(p)$ *when* $p|d$ *and* $L_1(p)$ *is* p^2*-modular,*

(iii) *if* L *is odd, with* $L_1(2) = \langle 4\eta \rangle$ *and* $M_1(2) = \langle 4\varepsilon \rangle$, *then* $\eta \equiv \varepsilon \bmod 4$.

The type of L_p is defined in [7, 91C]. It is not necessary to include the local invariant $dL_0(p) = dM_0(p)$ for $p|dD$ when $L_1(p)$ is p-modular since this is ensured by the local invariant $S_p V$ (see [6]). Theorem 4 can also be derived from some of the more general results in [1] or [7].

2. Proof of Theorem 1

The formula for $G_e(n, s, \Delta)$ was derived in [6] and the method of proof presented is basic to determining G_e and G_0 in other situations; only the local structure of L_2 changes since the structure of L_p, p odd, is the same as in [6]. Now assume L is odd. When $n = 3$ there are exactly eight possibilities for L_2; namely, $\langle 1, 1, \pm 1\rangle$, $\langle -1, -1, \pm 1\rangle$, $\langle 1, 1, \pm 3\rangle$, and $\langle -1, \pm 1, 3\rangle$ (since $\langle 1, 3\rangle = \langle -1, -3\rangle$, $\langle -1, -1\rangle = \langle 3, 3\rangle$, etc.). These eight lattices are distinguished by the eight possible values of the local invariants S_2V and dL_2. For $n > 3$, by [5, Theorem 1], L_2 is of the form $\langle 1, \dots, 1\rangle \perp N_2$ where N_2 has rank three and so is one of the eight possibilities above. As explained in detail in [6], for each of the e primes dividing d there are four possible local lattices L_p, and for each of the f primes dividing D there are two possible L_p. The choice for V_∞ is fixed by the signature. Therefore, Δ and Hilbert reciprocity uniquely determine the choice of L_2. For each set of choices of local lattices, an analogue of Theorem 1 in [6] ensures the existence of a corresponding $\mathbf{Z}$-lattice. Hence $G_0(n, s, \Delta) = 2^{2e+f}$. The above discussion includes a proof of Theorem 4 when Δ is odd; note that once V and Δ are fixed, so are S_2V and dL_2, and hence L_2 is uniquely determined by its type.

3. Proof of Theorem 2

We consider first even lattices so that $n \geq 3$ is odd, and modify the proof of Theorem 3 in [6]. Let

$$r = \sharp\{p : p | d \text{ and } S_pV = -1\}$$

and

$$t = \sharp\{p : p | D \text{ and } S_pV = -1\}.$$

By Hilbert reciprocity, $r + t \equiv c \bmod 2$ where $S_2VS_\infty V = (-1)^c$; we postpone calculating c until it is needed. For odd primes p, the local structure of L_p is essentially the same as in [6], since the extra 2 in the discriminant is a local unit and can be absorbed. However, there are now four possibilities for L_2; namely,

$$L_2 = H \perp \cdots \perp H \perp \langle 2\rangle \quad \text{when } D \equiv (-1)^{(s-1)/2} \bmod 8,$$
$$L_2 = H \perp \cdots \perp H \perp \langle -2\rangle \quad \text{when } D \equiv (-1)^{(s+1)/2} \bmod 8,$$
$$L_2 = H \perp \cdots \perp H \perp B_2 \perp \langle 2\rangle \quad \text{when } D \equiv 3(-1)^{(s+1)/2} \bmod 8,$$
$$L_2 = H \perp \cdots \perp H \perp B_2 \perp \langle -2\rangle \quad \text{when } D \equiv 3(-1)^{(s-1)/2} \bmod 8,$$

where B_2 is the binary anisotropic unimodular plane (since $B_2 \perp \langle 2\cdot 3\eta\rangle = H \perp \langle -2\eta\rangle$ for $\eta = \pm 1, \pm 3$). These four cases can be distinguished by the local discriminant dL_2. Moreover, the value of $S_2V = S_2L_2$ is now determined by D, n, and s; hence c is also a function of D, n, and s. The analogue of [6, Theorem 1] now holds when these four possibilities are incorporated. The analogue of [6, Theorem 2] then follows with the same proof (except the value of S_2V is changed). The same calculation as in [6, Theorem 3] then establishes Theorem 2, except that when $D = 1$, we get $G_e(n, s, \Delta) = 2^{e-1}(2^e + (-1)^c)$.

We must now compute $c \bmod 2$. In this case

$$L_2 = H \perp \cdots \perp H \perp \langle (-1)^{(s-1)/2} 2 \rangle$$

so that, using Hilbert symbols as in [6],

$$S_2 V = (-1)^{(n^2-1)/8} ((-1)^{(s-1)/2} 2, (-1)^{(n+1)/2})_2.$$

It follows that $c \equiv \frac{1}{8}(s^2 - 1) \bmod 2$, as required.

Now assume L is odd and $n \geq 3$ arbitrary. For fixed n it is easily seen that there are exactly eight possible lattices L_2 distinguished by the values of dL_2 and $S_2 V$; namely, $L_2 = \langle 1, \ldots, 1 \rangle \perp N_2$ where $N_2 = \langle \pm 1, 2 \rangle$, $\langle \pm 1, 2 \cdot 3 \rangle$, $\langle 3, \pm 2 \rangle$, or $\langle -3, \pm 2 \cdot 3 \rangle$. Thus $G_0(n, s, \Delta) = 2^{2e+f}$, as in the proof of Theorem 1. A proof of Theorem 4 for this situation is implicitly included above.

4. Proof of Theorem 3

Let $\Delta = (-1)^{(n-s)/2} 4d^2 D$ and assume first that L is even. The local structure for odd primes is the same as in [6]. When $n \geq 3$ is odd, there are two distinct possibilities for L_2 for each n, s, and D, namely,

$$L_2 = H \perp \cdots \perp H \perp \langle 4\eta \rangle \quad \text{with } \eta \equiv (-1)^{(s-1)/2} D \bmod 8,$$

and

$$L_2 = H \perp \cdots \perp H \perp B_2 \perp \langle 4\eta \rangle \quad \text{with } \eta \equiv (-1)^{(s+1)/2} 3D \bmod 8.$$

Modifying the arguments in [6], it follows that each of these L_2 gives 2^{2e+f-1} distinct genera when $D > 1$, so that $G_e(n, s, \Delta) = 2^{2e+f}$. For $D = 1$, each of these L_2 gives $2^{e-1}(2^e + (-1)^c)$ distinct genera, where $(-1)^c = S_2 V S_\infty V$. These two L_2 lie on spaces whose Hasse symbols $S_2 V$ have opposite signs (even when $D > 1$) and hence the two terms $2^{e-1}(-1)^c$ in the expressions counting the genera cancel. Thus $G_e(n, s, \Delta) = 2^{2e}$. This also establishes Theorem 4 in this case since, although $\eta \bmod 8$ is an invariant of L_2, the Hasse symbol $S_2 V$ and Δ already distinguish the possibilities.

Next consider $n \geq 4$ even. There are now ten possibilities for $L_2 = H \perp \cdots \perp H \perp N_2$ with N_2 one of the following rank four lattices:

(i) if $dN_2 = 4$, then $N_2 = H \perp 2H$, $B_2 \perp 2B_2$ or $H \perp \langle 2, -2 \rangle$,
(ii) if $dN_2 = -4.3$, then $N_2 = H \perp 2B_2$, $B_2 \perp 2H$ or $H \perp \langle 2, 2.3 \rangle$,
(iii) if $dN_2 = -4$, then $N_2 = H \perp \langle 2, 2 \rangle$ or $H \perp \langle -2, -2 \rangle$,
(iv) if $dN_2 = 4.3$, then $N_2 = H \perp \langle 2, -2.3 \rangle$ or $H \perp \langle -2, 2.3 \rangle$.

To see that this list is complete, observe that $H \perp \langle 2\eta \rangle = B_2 \perp \langle -2.3\eta \rangle$ for any $\eta \equiv 1 \bmod 2$. Thus when $D \equiv (-1)^{s/2} \bmod 4$ there are three possible L_2, while for $D \equiv -(-1)^{s/2} \bmod 4$ there are only two possible L_2. For each choice of L_2 a slightly modified form of Theorem 1 in [6] holds. When $D > 1$ it follows that $G_e(n, s, \Delta) = 3.2^{2e+f-1}$ for $D \equiv (-1)^{s/2} \bmod 4$, and $G_e(n, s, \Delta) = 2^{2e+f}$ for $D \equiv -(-1)^{s/2} \bmod 4$. Next assume $D = 1$. Then $\Delta \equiv \pm 4 \bmod 32$ and only the cases (i) and (iii) can occur. For (iii), where $s \equiv 2 \mod 4$, the two spaces V_2 and V_2' that occur have $S_2V_2 = -S_2V_2'$, so that the two terms $2^{e-1}(-1)^c$ cancel, and $G_e(n, s, \Delta) = 2^{2e}$. Finally, for (i), where $s \equiv 0 \mod 4$, we must consider $N_2 = H \perp 2H$, $B_2 \perp 2B_2$, or $H \perp \langle 2, -2\rangle$. The first and third $\mathbf{Z}_2$-lattices lie on the same hyperbolic space and hence have the same Hasse symbol $(-1)^{n(n+2)/8}$. Calculation shows that $B_2 \perp 2B_2$ lies on a space with the opposite Hasse symbol. It follows that $G_e(n, s, \Delta) = 2^{e-1}(3.2^e + (-1)^{s/4})$.

Finally, we consider the local structure of L_2 when L is an odd lattice. For each $n \geq 5$ there exist 32 distinct L_2 divided into three basic types according to the nature of the nonunimodular component $L_1(2)$ of L_2.

(i) There are eight distinct lattices of rank $n = 5$ when the nonunimodular component $L_1(2)$ is $2H$ or $2B_2$; namely, $\langle 1, 1, \pm 1\rangle \perp L_1(2)$ and $\langle -1, -1, \pm 1\rangle \perp L_1(2)$ (the equivalences $\langle \pm 3\rangle \perp 2B_2 = \langle \mp 1\rangle \perp 2H$ can be used to remove any $\langle \pm 3\rangle$ components). Similarly, when $n > 5$, there are eight distinct lattices obtained from the above by adjoining the rank $n-5$ lattice $\langle 1, \dots, 1\rangle$; these lattices are distinguished by the local invariants S_2V and dL_2. Note that when $n = 3$ only four lattices exist; one is excluded for each of the four possible values for dL_2. When $n = 4$ only six distinct lattices exist; the two excluded lattices have $dL_2 \equiv 4 \mod 16$. Note also, the lattices in this family do not diagonalize locally.

(ii) There are eight distinct lattices of rank $n \geq 3$ with a rank two nonunimodular component $L_1(2)$ that diagonalizes (adjoin $\langle 2\rangle$ to each of the lattices given at the end of the proof of Theorem 2). These eight lattices are distinguished by the invariants S_2V and dL_2.

(iii) There are sixteen distinct rank four lattices with $L_1(2) = \langle 4\eta\rangle$; namely, $\langle 1, 1, \pm 1, 4\eta\rangle$ and $\langle -1, -1, \pm 1, 4\eta\rangle$ where $\eta = \pm 1, \pm 3$. These lattices are distinguished by the invariants S_2V, dL_2, and $\eta \mod 4$. (If $L_1(2)$ is $\mathbf{Z}_2x$ and $\mathbf{Z}_2y$ in two Jordan splittings of L_2, then $x = ay + 4z$ for some $z \in L_2$ with $f(y, z) = 0$, and hence $f(x, x) \equiv f(y, y) \mod 16$, making $\eta \mod 4$ an invariant.) Higher rank lattices can be obtained by adjoining $\langle 1, \dots, 1\rangle$. However, when $n = 3$, four of the sixteen possible lattices do not exist, one for each of the four values of dL_2.

These 32 lattices account for the extra factor of 4 in $G_0(n, s, \Delta)$, $n \geq 5$. Only 24 of these lattices exist for $n = 3$, three for each value of dL_2 (which is determined by Δ). An adjustment is also needed when $n = 4$. This completes the proofs of Theorems 3 and 4.

5. Orthogonal splittings

For certain discriminants Δ an indefinite lattice L with $dL = \Delta$ diagonalizes if and only if L_2 has an orthogonal basis. This has been studied in [4]. The above theorems lead to further results. As in [4], let $\mathscr{D}$ (i) denote the set of discriminants of lattices L on spaces V, with Witt index at least $i = i(V)$, that diagonalize over $\mathbf{Z}$ whenever the localization L_2 diagonalizes. Also, let

$\mathscr{D}(\infty)$ denote the stable version consisting of discriminants where $dL \in \mathscr{D}(\infty)$ means the lattice $L \perp H^m$ diagonalizes for m sufficiently large, assuming L_2 diagonalizes.

Theorem 5. *Let* Δ *be cube-free and not divisible by any prime* $p \equiv 1 \mod 4$. *Then* $\Delta \in \mathscr{D}(\infty)$.

More precisely, for $e + f \geq 1$, it follows from Theorems 1, 2, 3, and 4 that $\pm d^2 D$, $\pm 2d^2 D \in \mathscr{D}(e+f)$ and $\pm 4d^2 D \in \mathscr{D}(e+f+1)$, provided each prime p dividing dD has $p \equiv 3 \mod 4$. The situation is far more complicated when primes $p \equiv 1 \mod 4$ divide Δ (see [4]). We sketch the argument for $\Delta = \pm 4d^2 D \in \mathscr{D}(e+f+1)$. It suffices to show that there exist 3.2^{2e+f} distinct classes of diagonalized lattices with rank n, signature s, and discriminant Δ, since one quarter of the lattices counted in $G_0(n, s, \Delta)$ do not diagonalize locally for $p = 2$. These lattices can be constructed as orthogonal sums by using as building blocks $\langle \pm 4 \rangle$ and $\langle 2, -2 \rangle$ for $p = 2$, $\langle \pm p \rangle$ for each $p | D$, $\langle \pm p^2 \rangle$, $\langle p, -p \rangle$ and either $\langle p, p \rangle$ or $\langle -p, -p \rangle$ for each $p | d$, and $\langle \pm 1 \rangle$ to fill out the required rank and signature (in some situations more care is needed). There are three choices when $p = 2$, two choices for each p dividing D, and four choices for each p dividing d. Since $p \equiv 3 \mod 4$ for each $p | dD$, so that -1 is not a local square, these 3.2^{2e+f} choices produce distinct sets of localizations of L, and hence cover all possible classes.

The following result follows from the fact that $G_0(n, s, \Delta)$ is independent of n and s. A related theorem for square-free discriminants was established in [5], but by a different method.

Theorem 6. *Let* L *be an indefinite odd lattice with rank* $n \geq 3$ *and cube-free discriminant* $\Delta \not\equiv 0 \mod 4$. *Then* L *has an orthogonal splitting*

$$L = \langle \pm 1, \ldots, \pm 1 \rangle \perp T,$$

where T *is an indefinite odd ternary lattice.*

Modified results hold when $\Delta \equiv 0 \mod 4$.

Theorem 7. *Let* L *be an indefinite even lattice with rank* $n \geq 3$ *and cube-free discriminant* Δ. *Then* L *has an orthogonal splitting*

$$L = H \perp \cdots \perp H \perp E_8 \perp \cdots \perp E_8 \perp T,$$

where T *is an even lattice of* rank t *with*

(i) $t = 3$ *if* $s \equiv 1, 3 \mod 8$,
(ii) $t = 5$ *if* $s \equiv 5 \mod 8$ (*with* $t = 3$ *if* $i(L) \geq 3$),
(iii) $t = 7$ *if* $s \equiv 7 \mod 8$ (*with* $t = 3$ *if* $i(L) \geq 2$),
(iv) $t = 4$ *if* $s \equiv 2, 4 \mod 8$,
(v) $t = 6$ *if* $s \equiv 6 \mod 8$ (*with* $t = 4$ *if* $i(L) \geq 3$),
(vi) $t = 8$ *if* $s \equiv 0 \mod 8$ (*with* $t = 4$ *if* $i(L) \geq 2$).

The key point in proving (i), for example, is that $G_e(n, s, \Delta) = G_e(3, 1, \Delta)$ when $s \equiv 1 \mod 8$, and hence there exist exactly enough indefinite ternary lattices T to cover all the possibilities for L; the number of hyperbolic planes H and even definite unimodular lattices E_8 is determined by the rank and signature of L. The proof is similar when $s \equiv 3 \mod 8$, except now T is definite. Similar proofs hold in the other cases.

Remarks. The values $t = 6$, 7, and 8 in (v), (iii), and (vi) cannot, in general, be improved (see [5] for 6 and 7, for $t = 8$ adjoin $\langle 2.3.7 \rangle$ to the example for $t = 7$). The other values for t are also best possible.

The splitting of indefinite lattices has also been considered by Gerstein [3] and Watson [8].

6. Square-free discriminants

We construct complete sets of representatives for the classes of even indefinite lattices with rank three, or four, and square-free discriminant. Let $D = p_1 \cdots p_f$ where the p_i are f distinct odd primes. Choose $\varepsilon_i = \pm 1$, $1 \leq i \leq f$, with $\prod_i \varepsilon_i = 1$; there exist 2^{f-1} distinct choices for the ε_i. First consider $D \equiv 3 \mod 4$. By Dirichlet's theorem, there exists a prime $q \equiv 1 \mod 4$ with $(q/p_i) = \varepsilon_i$, $1 \leq i \leq f$. It follows from quadratic reciprocity that $(-D/q) = 1$ and hence there exists an odd integer $a > 0$ with $Da^2 \equiv -1 \mod 4q$. Put $b = (Da^2 + 1)/4q$ and

$$B = B(\varepsilon_1, \dots, \varepsilon_n) = \begin{pmatrix} 2b & aD \\ aD & 2qD \end{pmatrix}.$$

Then B corresponds to an even definite lattice with $dB = D$. Also, locally, $B_p = \langle 2qD, 2q \rangle$ for $p|D$, and hence the 2^{f-1} lattices $B(\varepsilon_1, \dots, \varepsilon_n)$ are locally distinct (for some p). Since $G_e(4, 2, -D) = 2^{f-1}$, it follows that the 2^{f-1} lattices $H \perp B(\varepsilon_1, \dots, \varepsilon_n)$ form a complete set of representatives for the classes of even lattices with $n = 4$, $s = 2$, and $\Delta = -D$. Similarly, $G_e(3, 1, -2D) = 2^{f-1}$, and hence $\langle -2 \rangle \perp B(\varepsilon_1, \dots, \varepsilon_n)$ gives a complete set of representatives for the classes of even lattices with $n = 3$, $s = 1$, and $\Delta = -2D$.

Now let $D \equiv 1 \mod 4$. Choose ε_i as above, and a prime $q \equiv 1 \mod 4$ with $(q/p_i) = \varepsilon_i$, $1 \leq i \leq f$. Then $(D/q) = 1$ and there exists an odd integer $a > 0$ with $Da^2 \equiv 1 \mod 4q$. Put $c = (Da^2 - 1)/4q$ and

$$C = C(\varepsilon_1, \dots, \varepsilon_n) = \begin{pmatrix} 2c & aD \\ aD & 2qD \end{pmatrix}.$$

Then C is even and indefinite with $dC = -D$. The two sets of lattices $H \perp C(\varepsilon_1, \dots, \varepsilon_n)$ and $\langle 2 \rangle \perp C(\varepsilon_1, \dots, \varepsilon_n)$ form complete sets of representatives for the classes of even lattices with square-free discriminant for $n = 4$, $s = 0$ and for $n = 3$, $s = 1$, respectively.

The above constructions can be modified to get complete sets of representatives for indefinite odd lattices with $n = 3$ or 4 and square-free discriminant.

However, now all 2^f choices for the ε_i are needed, and the diagonal 2's in the matrix for B, or C, should be omitted.

REFERENCES

1. J. W. S. Cassels, *Rational quadratic forms*, Academic Press, London, 1978.
2. K. S. Chang, *Diskriminanten und Signaturen gerader quadratischer Formen*, Archiv. Math. **21** (1970), 59–65.
3. L. J. Gerstein, *Orthogonal splitting and class numbers of quadratic forms*, J. Number Theory **5** (1973), 332–338.
4. D. G. James, *Diagonalizable indefinite integral quadratic forms*, Acta Arith. **50** (1988), 309–314.
5. ——, *Orthogonal decompositions of indefinite quadratic forms*, Rocky Mountain J. Math. **19** (1989), 735–740.
6. ——, *Even quadratic forms with cube-free discriminant*, Proc. Amer. Math. Soc. **106** (1989), 73–79.
7. O. T. O'Meara, *Introduction to quadratic forms*, Springer-Verlag, New York, 1963.
8. G. L. Watson, *Integral quadratic forms*, Cambridge University Press, London, 1960.

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