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HOMOLOGICAL PROPERTIES OF COHERENT SEMILOCAL RINGS

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ABSTRACT. W. V. Vasconcelos and M. Auslander have studied the homological properties of coherent (noetherian) rings. In [3] some theorems for regular local rings were generalized to noetherian semilocal rings. The main aim of this paper is to discuss coherent semilocal rings.

1. HOMOLOGICAL DIMENSIONS OF COHERENT SEMILOCAL RINGS

Let R be a commutative ring with identity element, and let J be the Jacobson radical of R. The concepts and the notations that are used in this paper are consistent with those in [14].

Definition. Let A be an R-module. A normal A-sequence is an ordered sequence $u_1, u_2, \dots, u_n$ in J such that u_1 is not a zero divisor in A and, for $i > 1$, each u_i is not a zero divisor in $A/(u_1, \dots, u_{i-1})A$. We write $\operatorname{cod}_R(A) = n$ if there exists a normal A-sequence with n terms but no normal sequence with more than n terms.

Theorem 1.1. *Let R be a coherent semilocal ring such that J is finitely generated, and let A be a finitely presented R-module. Then*

(i) $\text{Tor-dim}\, R = pd_R(R/J)$:
(ii) *If $u_1, u_2, \dots, u_n$ is a normal A-sequence, then*

$$pd_R(A/u(u_1, \dots, u_n)A) = pd_R(A) + n.$$

Proof. (i) Because R/J is a finitely presented R-module and R a coherent ring, $pd_R(R/J) = \text{flat-dim}_R(R/J) \le \text{Tor-dim}\, R$ by [2, Proposition 2]. Assume that $pd_R(R/J) = n < \infty$. If A is a finitely presented R-module, then $\operatorname{Tor}^R_{n+1}(A, R/J) = 0$. Hence $\operatorname{Tor}^{R_M}_{n+1}(A_M, R_M/J_M) = 0$, where M is any maximal ideal of R. Since R is a semilocal ring, $J_M = M_M$. Thus

$$\operatorname{Tor}^{R_M}_{n+1}(A_M, R_M/M_M) = 0,$$

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and by [14, Lemma 5.11], $pd_{R_M}(A_M) \leq n$. From [6, (3.E)] we have

$$\begin{aligned}\text{flat-dim}_R(A) &= \sup\{\text{flat-dim}_{R_M}(A_M) | M \text{ is a maximal ideal of } R\} \\ &= \sup\{pd_{R_M}(A_M) | M \text{ is a maximal ideal of } R\},\end{aligned}$$

which shows that $\text{flat-dim}_R(A) \leq n$. On the other hand,

$$\text{Tor-dim}\, R = \sup\{\text{flat-dim}_R(A) | A \text{ is a finitely presented } R\text{-module}\},$$

hence

$$\text{Tor-dim}\, R = pd_R(R/J).$$

(ii) The statement can be shown in a proof similar to that of [10, Theorem 20, p. 196].

The Auslander–Buchsbaum Theorem affirms that if A is a finitely generated module over a regular local ring R, then $\text{pd}_R(A) + \text{cod}_R(A) = \text{gl.dim}\, R$. We give a similar result over a coherent local ring as follows.

Theorem 1.2. *Let R be a coherent local ring such that $\text{Tor-dim}\, R < \infty$, and let J be finitely generated. If A is a finitely presented noetherian R-module, then* $pd_R(A) + \text{cod}_R(A) = \text{Tor-dim}\, R$.

Before proving Theorem 1.2, we first give

Lemma. *Let R be a coherent local ring such that $\text{Tor-dim}\, R < \infty$, and let J be finitely generated. Suppose that A is a finitely presented noetherian R-module. Then*

(i) *If the only submodule B of A such that $JB = 0$ is the zero submodule, then J contains an element which is not a zero divisor in A.*

(ii) *If there exists a submodule $B \neq 0$ of A such that $JB = 0$, then* $pd_R(A) = \text{Tor-dim}\, R$.

Proof. (i) See the proof of [9, Theorem 17, p. 186].

(ii) We may assume $B = (x)$. The homomorphism $R \to B$, in which $r \in R$ is mapped into rx, has kernel J, hence $B \cong R/J = F$, and then we have an exact sequence $0 \to F \to A \to A/B \to 0$. Assume $\text{Tor-dim}\, R = p$, then Tor^R_{P+1} vanishes identically, and therefore $0 \to \text{Tor}^R_P(F, F) \to \text{Tor}^R_P(A, F)$ is exact. But $\text{Tor}^R_P(F, F) \neq 0$, so $\text{Tor}^R_P(A, F) \neq 0$, which shows that $\text{flat-dim}_R(A) \geq p$. Since A is finitely presented, $\text{flat-dim}_R(A) = pd_R(A)$. Hence $pd_R(A) = \text{Tor-dim}\, R$.

Proof of Theorem 1.2. It is clear by Theorem 1.1 that $\text{cod}_R(A) \leq \text{Tor-dim}\, R < \infty$. Assume $\text{cod}_R(A) = s$. Then there exists a normal A-sequence $u_1, \ldots, u_s$. By Theorem 1.1, we have

$$\text{cod}_R(A) + pd_R(A) = s + pd_R(A) = pd_R(A/(u_1, \ldots, u_s)A).$$

Now, we need only to show that $pd_R(A/(u_1, \ldots, u_s)A) = \text{Tor-dim}\, R$ and, by the Lemma, this will follow if we can show that J annihilates some nonzero submodule of $A/(u_1, \dots, u_s)A$. Assume the contrary. Since A is a finitely presented noetherian R-module, so is $A/(u_1, \dots, u_s)A$. By the Lemma there exists $u_{s+1} \in J$ such that u_{s+1} is not a zero divisor in $A/(u_1, \dots, u_s)A$. But this means that $u_1, \dots, u_s, u_{s+1}$ is a normal A-sequence, and therefore $\mathrm{cod}_R(A) \geq s+1$, which contradicts the above hypothesis. Hence $\mathrm{cod}_R(A) + pd_R(A) = \text{Tor-}\dim R$.

2. The structure of coherent semilocal rings

[14, Theorem 2.2] says that a local ring of global dimension 2 is either a regular local ring, a valuation domain, or an umbrella ring. Here we discuss coherent semilocal rings, and similar results will be obtained.

Theorem 2. *Let R be an indecomposable semilocal ring. If R has the property that* $\mathrm{gl.dim}\, R_M \leq r$ *for each maximal ideal M, then* $\mathrm{gl.dim}\, R \leq r^{(*)}$. *Furthermore, if* $\mathrm{gl.dim}\, R = 2$, *then*

(i) *R is a coherent GCD domain.*
(ii) *Every ideal of R is countably generated.*
(iii) *R must be one of the following: a $(1, 2, 3)$-ring, a $(2, 2, 0)$-ring, or a $(2, 2, 3)$-ring.*

Remark. We say a ring R is an (a, b, c)-ring provided that $\text{Tor-}\dim R = a$, $\mathrm{gl.dim}\, R = b$, and $\mathrm{f.p.dim}\, R = c$ [8]. Before proving Theorem 2.1, we first give the following:

Proposition 2.1. *Let R be an indecomposable coherent semilocal ring. Then R is a GCD domain if and only if $pd_R(a, b) \leq 1$, $\forall a, b \in R$.*

Proof. Assume R is a GCD domain. $\forall a, b \in R$, if (a, b) is a principal ideal, then $(a, b) \cong R$, and then $pd_R(A, b) = 0$. From now on, we assume (a, b) is not principal. Consider the exact sequence

$$0 \to K \to R^{(2)} \xrightarrow{f} (a, b) \to 0, \tag{1}$$

where $f(1, 0) = a$, $f(0, 1) = b$, and $K = \mathrm{Ker}\, f$. Obviously $K \cong (a) \cap (b)$. Let $[a, b] = d$. Then there exists a', b' in R such that $a = da'$, $b = db'$, and $[a', b'] = 1$. Write $c = da'b'$. Then $(c) \subseteq (a) \cap (b)$. $\forall x \in (a) \cap (b)$, there exist r, s in R such that $x = ra = sb$, and then $ra'd = sb'd$. Since R is a domain, $ra' = sb'$. But $[a', b'] = 1$; thus, $b' | r$. That is, there exists r' in R such that $r = r'b'$. It follows that $x = ra = ra'd = r'b'a'd = r'c$; that is, $x \in (c)$, and then $(a) \cap (b) = (c)$. Hence $K \cong (c)$ and so K is a projective R-module, which shows that $pd_R(a, b) \leq 1$.

For the converse, $\forall 0 \neq a \in R$, from the exact sequence $0 \to \mathrm{Ann}(a) \to R \to (a) \to 0$ and $pd_R(a) \leq 1$, we know that $\mathrm{Ann}(a)$ is finitely generated projective ideal, and then $\mathrm{Ann}(a)$ is free by [3, Corollary 1.2]. But $a \neq 0$, thus $\mathrm{Ann}(a) = 0$, which means that R is a domain. Furthermore, $\forall a, b \in R$, assume (a, b) is not a principal ideal. Since $pd_R(a, b) \leq 1$ and R is a coherent ring, K is projective in the exact sequence (1), and then free. But R is a domain and $K \cong (a) \cap (b)$; thus, $(a) \cap (b)$ is principal, which shows that K is a free module with rank 1. Let K be generated by (a_1, b_1). Since $f(-b, a) = -ba + ab = 0$, $(-b, a) \in K$; that is, there exists d in R such that $(-b, a) = d(a_1, b_1)$. Hence $a = da_1$, $b = -db_1$. It is easy to verify that $d = [a, b]$. Consequently, R is a GCD domain.

Proposition 2.2. *Let R be a semilocal ring and A an R-module. Then*

(i) *A is finitely (countably) generated if and only if A_M is a finitely (countably) generated R-module for each maximal ideal M of R.*

(ii) *R is a noetherian (coherent) ring if and only if R_M is noetherian for each maximal ideal M of R.*

Proof. (i) We need only prove the sufficiency. Assume all maximal ideals of R are $M_1, \dots, M_t$ and A_{M_i} be finitely generated R_{M_i}-module for each M_i. Let $X_i = \{x_{it}/1 | i_t \in T_i\}$ be a finitely generated set of A_{M_i}. Then $X = \biguplus_{i=1}^t X_i$ and $T = \biguplus_{i=1}^t T_i$ are finite sets, where $\biguplus$ means disjoint union. Define $f: R^{(T)} \to A$ by $f(e_{i_t}) = x_{i_t}$, where e_{i_t} is that element having 1 in the i_tth coordinate and 0 elsewhere. Obviously, f is a R-homomorphism, and $f_{M_{i_t}} : R_{M_i}^{(T)} \to A_{M_i}$ is surjective. Consequently, f is surjective, showing that A is finitely generated.

Similarly for the countably generated case.

(ii) follows immediately from (i).

Proof of Theorem 2.1. If a semilocal ring R has the property $\text{gl.dim}\, R_M \le r$ for each maximal ideal M then $\text{gl.dim}\, R \le r$. This follows from the descent of projectivity for faithfully flat morphisms (in the case $R \to R_{M_1} \times \cdots \times R_{M_n}$) of Gruson and Raynaud [11, Example 3.1.4 (1), p. 82]. Now, assume $\text{gl.dim}\, R = 2$.

(i) Since $\text{gl.dim}\, R_M \le 2$ for each maximal ideal M of R, R_M is a coherent GCD domain [14, Theorem 2.2]. By Proposition 2.2, R is a coherent ring. $\forall a, b \in R$, by Proposition 2.1, $pd_{R_M}(a, b)_M \le 1$. Hence $pd_R(a, b) = \text{flat-dim}_R(a, b) = \sup_M\{\text{flat-dim}_{R_M}(a, b)_M\} = \sup_M\{pd_{R_M}(a, b)_M\} \le 1$, and then R is a GCD domain.

(ii) Since $\text{gl.dim}\, R_M \le 2$ for each maximal ideal M of R, each ideal of R_M is countably generated [14, Theorem 4.8], and then so is each ideal of R, by Proposition 2.2.

(iii) First, we have that $\text{f.p.dim}\, R \ne 2$, or else there exists some ideal I of R such that $\text{f.p.dim}_R(I) = 1$ [8, Corollary 2.9]. Hence for each maximal ideal M of R. $\text{f.p.dim}_{R_M}(I_M) \le 1$, but $\text{f.p.dim}\, R_M \ne 2$, so $\text{f.p.dim}_{R_M}(I_M) = 0$ which means that I_M is finitely presented. By Proposition 2.2 (i), $\text{f.p.dim}_R(I) = 0$; thus we get a contradiction.

From [8, Theorem 3.4] we have that

$$\text{gl.dim}\, R = \sup\{\text{Tor-dim}\, R,\ \text{f.p.dim}\, R - 1\}$$

for R is coherent. If $\mathrm{gl.dim}\, R = 2$, then either $\text{Tor-dim}\, R = 1$ and $\text{f.p.dim}\, R = 3$, $\text{Tor-dim}\, R = 2$ and $\text{f.p.dim}\, R = 2$, or $\text{Tor-dim}\, R = 2$ and $\text{f.p.dim}\, R = 3$, and the theorem is proved.

Finally, we give

Theorem 2.2. *Let R be an indecomposable semilocal ring of global dimension 2. Then*

(i) *R is a $(1, 2, 3)$-ring if and only if each maximal ideal of R is either principal, or nonfinitely generated if and only if R is a Bezout domain.*
(ii) *R is a $(2, 2, 0)$-ring if and only if R satisfies the ascending chain condition on principal ideals if and only if R is noetherian.*
(iii) *In addition, if J is finitely generated, then R is a $(2, 2, 3)$-ring if and only if there exists a maximal ideal which is not principal, and R is not completely integrally closed.*

Remarks. A domain R with quotient field K is completely integrally closed if, for a and u in K with $a \neq 0$, $au^n \in R$ for all n implies $u \in R$.

Proof. (i) By Theorem 2.1, R is a coherent ring. Assume R is a $(1, 2, 3)$-ring; then R is a Bezout domain [2, Theorem 4], and then each maximal ideal is either principal or nonfinitely generated.

For the converse, if each maximal ideal M of R is either principal, or nonfinitely generated, then so is M_M, by Proposition 2.2. Hence, by [14, Theorem 2.2], R_M is a valuation domain, which implies that $\text{Tor-dim}\, R = \sup_M\{\text{Tor-dim}\, R_M\} \leq 1$. But $\mathrm{gl.dim}\, R = 2$, by Theorem 2.1 R is a $(1, 2, 3)$-ring.

(ii) It is clear that R is a $(2, 2, 0)$-ring if and only if R is a noetherian ring. Now, assume R satisfies the ascending chain condition on principal ideals. One wishes to prove that R is a noetherian ring. By [14, Theorem 8] it is enough to show that each prime ideal of R is finitely generated. Let P be a prime ideal of R. By [2, Corollary 4], R is UFD. Hence, when P is maximal, there exists an element d in P such that $P = (d)$. When P is not maximal, by Theorem 2.1, R is a coherent GCD domain with global dimension 2, again by the proof of [14, (7), p. 21], P is a directed union of principal ideals. Assume $S = \{(a) | a \in P\}$, and apply Zorn's Lemma to S. There exists a maximal element (b) in S. For any element a in P, there exists c in P such that $b \in (c)$ and $a \in (c)$. Hence $(b) = (c)$, which shows that $P = (b)$.

(iii) For the sufficiency, it follows from (i) that R is not a $(1, 2, 3)$-ring. Observe that R is a GCD domain, and then R is integrally closed [4, Theorem 50]. If R is a $(2, 2, 0)$-ring, then R is an integrally closed noetherian domain, and then R is completely integrally closed by [4, Example 12, p. 53], a contradiction. Consequently, R is a $(2, 2, 3)$-ring by Theorem 2.1.

For necessity, assume R is a $(2, 2, 3)$-ring. By (i) there exists a maximal ideal of R which is not principal, so we need only to show that R is not completely integrally closed. If this were not so, then R_M is completely integrally closed for each maximal ideal M of R. Observe that $\text{gl.dim}\, R_M \leq 2$. When $\text{gl.dim}\, R_M \leq 1$, R_M is noetherian. When $\text{gl.dim}\, R_M = 2$, if M is a principal ideal, by (i) R_M is a $(1, 2, 3)$-ring which shows R_M is a completely integrally closed valuation domain, and then $\text{Krull-dim}\, R_M \leq 1$ [4, Example 12, p. 53]; hence, R_M is noetherian; if M is not principal, then R_M is also noetherian [13, theorem]. Consequently, for any maximal ideal M of R, R_M is noetherian. By Proposition 2.2, R is noetherian, and we get a contradication; therefore, R is not completely integrally closed.

Note. It is suggested by the referee that if a semilocal ring R has the property that $\text{gl.dim}\, R_P \leq r$, for each maximal ideal P, then $\text{gl.dim}\, R \leq r$. The authors express their thanks.

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