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ON LINEAR GROUPS OVER FINITE FIELDS

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ABSTRACT. Let G be a finite group with an Abelian Sylow p-subgroup P $(p > 5)$, and F, a finite field of characteristic p. Set $H = O^{p'}(G)$. If G has a faithful FG-module M such that $\dim_F M < p-2$, then one of the following is true:

(a) P is normal in G,
(b) $H/Z(H) \approx \oplus_{i \leq t} L_2(p^{n_i})$, where n_i and t are positive integers and $2t < p-2$,
(c) $p = 7$ or 11 and $H \approx 2.A_7$ or J_1, respectively, $\dim_F M \geq p-4$.

In 1963, R. Brauer raised forty-three important problems on group and representation theories [2]. The fortieth problem is as follows:

Brauer Problem 40. Determine the linear groups G of small degrees over a finite field F.

Let P be a Sylow p-subgroup of G, where p is the characteristic of F. About 25 years ago, Feit began the study of this problem for $|P| = p$ [5], [6]. His results generalized theorems of Brauer [1] and Tuan [4] on ordinary representation. Recently, Blau [11] gave very nice results on the problem when P is cyclic. Since $SL(2, p^n)$ has all d with $2 \leq d \leq p-1$ as the dimension of an irreducible representation over a suitably large finite field of characteristic p, it is in general rather difficult to determine the group structure of a linear group over a finite field. In the present paper, under the assumption that P is Abelian, we will characterize the linear groups of degree less than $p-2$ in terms of group theoretical properties. Our results extend the main theorem of Ferguson [10].

All groups in this paper are assumed to be finite, and the notation and terminology are standard and follow that of [7] and [14].

Linear groups of degree at most 4 have been determined [2], [15]. Therefore we will assume in the following that p is greater than 5.

Lemma 1. *Let G be a finite simple group of Lie type. If the characteristic of G is p with $p > 5$, and the Sylow p-subgroup of G is Abelian, then G is isomorphic to $L_2(p^n)$ forsome integer $n \geq 1$.*

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Proof. It is an easy consequence of the Chevalley commutator identities.

Lemma 2. *Suppose G is a finite p-nilpotent group and V is a faithful FG-module, where F is a finite field of characteristic p. Let P be a Sylow p-subgroup of G. If $\dim_F V < p-1$, then P is normal in G.*

Proof. If the lemma is not true, let (G, V) be a counterexample such that $|G| + \dim_F V$ is minimal. Since P is not normal in G, we can choose an element $y \in P \setminus O_p(G)$ such that $y^p \in O_p(G)$ and $YO_p(G)$ is normal in P, where $Y = \langle y \rangle$. Set $T = YO_{p'}(G)$. If T is a proper subgroup of G, then $(T, V|_T)$ satisfies the condition of the lemma. By the minimum of (G, V), Y is normal in T. It follows that $YO_p(G)$ is normal in $PO_{p'}(G) = G$, contrary to the choice of y. So $T = G$. Let $V_1, V_2, \dots, V_s$ be all composite factors of V and N_i be the kernel $\operatorname{Ker} V_i$ of V_i. Then the intersection $\bigcap_i N_i$ is a subgroup of $O_p(G)$. If $s > 1$, by the minimum of (G, V), YN_i is normal in G. $[y, O_{p'}(G)] \le N_i$, so $[y, O_{p'}(G)] \le \bigcap_i N_i \le O_p(G)$. Therefore the Sylow p-subgroup of G is a normal subgroup, contrary to the assumption on G. So V is irreducible, $O_p(G) = 1$, and P is of order p. Since $\dim_F V < p-1$ and $y - 1 \in J(FP)$, the radical of FP, $V(y-1)^{p-2} = 0$. By Hall–Higman Theorem B, $O_p(G) \ne 1$, a contradiction. The contradiction proves the lemma.

Theorem 3. *Let G be a finite group with an Abelian Sylow p-subgroup P $(p > 5)$ and F an arbitrary finite field of characteristic p. Set $H = O^{p'}(G)$. If G has a faithful FG-module M such that $\dim_F M < p-2$, then one of the following must hold:*

(a) *P is normal in G,*
(b) *$H/Z(H) \approx \oplus_{i \le t} L_2(p^{n_i})$, where n_i and t are positive integers, $2t < p-2$, and $Z(H)$ is the center of H,*
(c) *$p = 7$ or 11 and $H \approx 2.A_7$ or J_1, respectively, $\dim_F M \ge p-4$.*

Proof. Suppose the theorem is not true, and let G be a counterexample of minimal order. Then

1. $PO_{p'}(G)$ is p-closed; i.e, P is normal in $PO_{p'}(G)$.
   P is normal in $PO_{p'}(G)$ by Lemma 2.
2. $G = H = O^{p'}(G)$, $Z(G) = O_{p'}(G)O_p(G)$.
   Clearly $H = \langle P^x | x \in G \rangle$ and by the minimality of G, $H = G$. By (1) $P \le C_G(O_{p'}(G))$. Hence, $H \le C_G(O_{p'}(G))$. Now $H = G$ yields $O_{p'}(G) \le Z(G)$. Similarly, since P is Abelian, $O_p(G) \le Z(G)$.
3. $G = F^*(G)$, the generalized Fitting subgroup of G, and G is perfect; i.e, $G' = G$.
   By (2) and the definition of $F^*(G)$, $\overline{F^*(G)} = F^*(G)/Z(G) = \overline{N}_1 \times \overline{N}_2 \times \cdots \times \overline{N}_s$, where $Z(G) \le N_i$ and $\overline{N}_i$ is non-Abelian simple and contains p as a prime divisor of its order. Let y be an arbitrary element of P. $P \cap F^*(G) = P_1P_2 \cdots P_s$, where P_i is a Sylow p-subgroup of N_i. Since P is Abelian, $[y, P_i] = 1$. For each N_i, $\overline{N}_i^y$ is also normal in $\overline{F^*(G)}$ and P_i is

contained in $N_i \cap N_i^y$, so $\overline{N}_i^y = \overline{N}_i$. By [16], y induces an inner automorphism of $\overline{N}_i$, so there exists a p-element x_i of N_i such that $\overline{yx_i}$ centralizes $\overline{N}_i$. Then yx_i centralizes N_i. It follows that $y(x_i x_2 \cdots x_s) \in C_G F^*(G)) \leq F^*(G)$, $y \in F^*(G)$. By (2), $F^*(G) = G$. Now it is easy to see that G is perfect by the minimum of G.

4. $G/Z(G)$ is non-Abelian simple.

If $G/Z(G)$ is not simple, then $G/Z(G) \approx \overline{M}_1 \times \overline{M}_2 \times \cdots \times \overline{M}_t$, $t \geq 2$, M_i contains $Z(G)$ as a subgroup, and $\overline{M}_i$ is non-Abelian simple. By the minimum of G, the theorem is true for M_i. If there is i, say $i = 1$, such that $M_i' \approx 2.A_7$ for $p = 7$ then, by Blau [11], $\dim_F M \geq 4$. Hence M_2' is isomorphic to either $2.A_7$ or $L_2(7^n)$. If M_2' is isomorphic to $2.A_7$, then $(M_1M_2)'$ is a homomorphism image of $2.A_7 \times 2.A_7$. Every nontrivial $F(2.A_7 \times 2.A_7)$-module U is of dimension at least $4 + 4 = 8$. It follows that $7 - 3 = 4 \geq \dim_F M \geq 4 + 4 = 8$, which is absurd. If $\overline{M}_2$ is isomorphic to $L_2(7^n)$, then that will lead to a similar contradiction on dimensions. Similarly, there exists no i such that $M_i' \approx J_1$ with $p = 11$. Therefore $M_i/Z(G)$ is isomorphic to $L_2(p^{n_i})$ for some positive integer n_i. Since $\dim_F(M|_{M_i}) \geq 2$, $2t < p - 2$. This shows that the theorem is true for G. This contradicts the assumption on G.

5. P is not cyclic .

This follows obviously from (4) and [11].

6. Last contradiction.

Set $\overline{G} = G/Z(G)$. If $\overline{G}$ is isomorphic to $A_n (n \geq 5)$, then by (5) $2p \leq n \leq p^2 - 1$. There is a subgroup B_0 of G such that $Z(G) \leq B_0$ and $\overline{B}_0$ is isomorphic to $A_p \times A_p$. Notice that $p > 5$, $\dim_F M|_{B_0} \geq 2(p - 3)$. $p - 3 \geq 2(p - 3)$. This, too, is absurd.

If $\overline{G}$ is isomorphic to $G(q)$, a simple group of Lie type, and q is a power of a prime r, then by Lemma 1 p is not equal to r. If $\overline{G}$ is isomorphic to $PSL(n, q)$, then by [12], with $p > 5$, $p - 3 \geq (q - 1)/d$ for $n = 2$ or $p - 3 \geq q^{n-1} - 1$ for $n > 2$, where $d = (2, q - 1)$. If $n = 2$, $p \geq (q - 1)/d + 3 = (q + 3d - 1)/d$. So $p > (q - 1)/d$. Since p is a prime divisor of $|\overline{G}|$, $p|(q + 1)/d$. Hence $(q + 3d - 1)/d \leq p \leq (q + 1)/d$, which is absurd. If $n > 2$, then $p \geq q^{n-1} + 2$. Since $p|(q^i - 1)$ for some positive integer $i \leq n$, $p|(q^n - 1)/(q - 1)$. Suppose $(q^n - 1)/(q - 1) = tp$. If $t \geq 2$, then $q \leq t(q - 1)$. $q^n - 1 = tp(q - 1) \geq q(q^{n-1} + 2)$, which is absurd. So $t = 1$, $p = (q^n - 1)/(q - 1)$. Then P is of order p, contrary to (5). Similarly, G is not isomorphic to any one of the following groups: $PSP(2n, q)$, $PSU(n, q)$, $PSO^+(2n, q)'$, $PSO^-(2n, q)'$, $PSO(2n+1, q)$, $G_2(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$.

If $\overline{G}$ is isomorphic to ${}^2F_4(2)'$, then $p = 13$. Hence P is of order 13, contrary to (5). If $\overline{G}$ is isomorphic to ${}^2F_4(q)$, $q = 2^{2m+1}$, $m \geq 1$, then by [12] $p - 3 \geq (q/2)^{1/2} q^4 (q - 1)$, $p \geq q^4 + 1$. The order of ${}^2F_4(q)$ is $q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$, so $p|(q^6+1)$. Since $q^6+1 = (q^2+1)(q^4-q^2+1)$,

$p \leq q^4 - q^2 + 1$, contradicting $p \geq q^4 + 1$. By a similar argument, we can show that $\overline{G}$ is not isomorphic to any one of the following groups: ${}^2E_6(q)$, ${}^3D_4(q)$, $Sz(q)$, ${}^2G_2(q)$.

By the classification of finite simple groups, $\overline{G}$ is isomorphic to a sporadic simple group. It is easy to check by the Atlas [14] that $\overline{G}$ is isomorphic to Co_1, B, or Th with $|P| = 49$ or F_1 with $|P| = 121$. There exists an extra-special 2-subgroup of order 2^{1+8} in each of the four simple groups. It follows that $\dim_F M \geq 2^4$. So $11 - 3 \geq p - 3 \geq 16$, which is absurd. The contradiction proves the theorem.

Corollary 4. *Suppose G is a finite group with an Abelian Sylow p-subgroup $P(p > 11)$. If G has a faithful FG-module of degree at most $p - 3$ over a field F of characteristic p, then either P is normal in G or $O^{p'}(G)/Z(O^{p'}(G))$ is isomorphic to $\oplus_{i \leq t} L_2(p^{n_i})$, $n_i \geq 1$, $2t < p - 2$.*

This result is similar to that of Ferguson [10].

Remark. If F is of characteristic zero, that P is Abelian directly follows that $\dim_F M < p - 2$. But in the modular case, if we did not assume that P is Abelian, there would be many simple groups added to the list, which would make Theorem 3 less meaningful.

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