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THE BEHAVIOR OF THE ZERO-BALANCED HYPERGEOMETRIC SERIES ${}_pF_{p-1}$ NEAR THE BOUNDARY OF ITS CONVERGENCE REGION

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ABSTRACT. For a zero-balanced generalized hypergeometric function ${}_pF_{p-1}(z)$, the authors prove a formula exhibiting its behavior near the boundary point $z = 1$ of the region of convergence of the series defining it. The result established here provides an interesting extension of a formula which appeared in one of Ramanujan's celebrated Notebooks; it also serves to solve the problem posed by R. J. Evans [5].

In many areas of applications involving various hypergeometric functions of one, two, and more variables, one finds the need to know the behaviors of these functions near the boundaries of the regions of convergence of the series defining them. Several results of this type have appeared in the literature (cf., e.g., [11]–[16]); some of these results were used in solving various boundary value problems involving the Euler–Darboux equation and in the study of certain operators of fractional calculus (cf. [10] and [18]).

The behavior of the Gaussian hypergeometric function ${}_2F_1(z)$ near $z = 1$ is given by a well-known analytic continuation formula [4, p. 108, Equation 2.10(1)]. An asymptotic formula exhibiting the behavior of a zero-balanced Clausenian hypergeometric function ${}_3F_2(z)$ near $z = 1$ was stated (without proof) by Ramanujan ([9, Chapter 11, Entry 24, Corollary 2]; see also [2, p. 295, Corollary 2]). A generalized hypergeometric function ${}_{p+1}F_p(z)$ defined, in terms of the Pochhammer symbol $(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda)$, by

$$(1)\qquad {}_{p+1}F_p[(a_{p+1});(b_p);z] = {}_{p+1}F_p\left[\begin{matrix}(a_{p+1}); \\ (b_p);\end{matrix}\; z\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p+1}(a_j)_n}{\prod_{j=1}^{p}(b_j)_n}\frac{z^n}{n!},$$

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is said to be ω-balanced if

$$\omega = \sum_{j=1}^{p} b_j - \sum_{j=1}^{p+1} a_j \tag{2}$$

is an integer. It may be recalled that the series in (1) converges absolutely when $|z| < 1$, or when $|z| = 1$ and $\mathrm{Re}(\omega) > 0$, provided that no zeros appear in the denominator. Here, for the sake of brevity, (a_p) abbreviates the array of p parameters $a_1, \ldots, a_p$, with similar interpretations for (b_q), etc.

Several different proofs of Ramanujan's formula for the behavior of a zero-balanced ${}_3F_2(z)$ near $z = 1$ were given, among others, by Saigo [12, p. 31], by Evans and Stanton [6, p. 1019], and by Bühring [3], who also generalized Ramanujan's formula for s-balanced hypergeometric ${}_3F_2$ series. Recently, while referring to the difficulty in the proof of Ramanujan's formula by Evans and Stanton [6], Evans [5, p. 553] posed the problem of finding analogues of Ramanujan's formula for ${}_4F_3$ and other higher-order hypergeometric series. In an attempt to solve this problem of Evans [5], we establish a formula exhibiting the behavior of a zero-balanced hypergeometric series ${}_pF_{p-1}(z)$ near the boundary point $z = 1$ of its region of convergence. A markedly different approach than ours to such problems involving ${}_4F_3$ and other higher-order hypergeometric series was made by Nørlund [8] and, more recently, by Marichev and Kalla [7].

We begin by considering the Kampé de Fériet series (cf. [1, p. 150]; see also [17, p. 27, Equation 1.3(28) *et seq.*]):

(3)

$$\Omega(x, y) = F^{0:p-1;2}_{1:p-3;0}\left[\begin{matrix} \rule{3cm}{0.4pt} : & (\alpha_{p-1}); & \beta_{p-2}-\alpha_p, \beta_{p-1}-\alpha_p; \\ \beta_{p-2}+\beta_{p-1}-\alpha_p : & (\beta_{p-3}); & \rule{3cm}{0.4pt}; \end{matrix} \; x, y\right] \quad (p \geqq 3)$$

where, for convergence, $\max\{|x|, |y|\} < 1$. Writing $\Omega(x, y)$ as an infinite series of powers of y, if we separate the first term, we get

(4)

$$\Omega(x, y) = {}_{p-1}F_{p-2}\left[\begin{matrix} (\alpha_{p-1}); \\ (\beta_{p-3}), \beta_{p-2}+\beta_{p-1}-\alpha_p; \end{matrix} \; x\right] + \frac{(\beta_{p-2}-\alpha_p)(\beta_{p-1}-\alpha_p)}{\beta_{p-2}+\beta_{p-1}-\alpha_p} y$$

$$\cdot F^{0:p-1;3}_{1:p-3;1}\left[\begin{matrix} \rule{3cm}{0.4pt} : & (\alpha_{p-1}); & \beta_{p-2}-\alpha_p+1, \beta_{p-1}-\alpha_p+1, 1; \\ \beta_{p-2}+\beta_{p-1}-\alpha_p+1 : & (\beta_{p-3}); & \rule{3cm}{0.4pt} \; 2; \end{matrix} \; x, y\right].$$

On the other hand, upon writing $\Omega(x, y)$ as an infinite series of powers of x, if we apply the Gaussian summation theorem [4, p. 104, Equation 2.8(46)]:

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{5}$$

$$(\mathrm{Re}(c-a-b) > 0; c \neq 0, -1, -2, \ldots),$$

we find directly from (3) that

$$(6)\quad \Omega(x,1)=\frac{\Gamma(\alpha_p)\Gamma(\beta_{p-2}+\beta_{p-1}-\alpha_p)}{\Gamma(\beta_{p-2})\Gamma(\beta_{p-1})}\,{}_pF_{p-1}\left[\begin{matrix}(\alpha_p);\\(\beta_{p-1});\end{matrix}\; x\right]\quad(\mathrm{Re}(\alpha_p)>0).$$

Combining (4) and (6) appropriately, we have

(7)

$$\begin{aligned}&{}_pF_{p-1}\left[\begin{matrix}(\alpha_p);\\(\beta_{p-1});\end{matrix}\; 1-\rho\right]\\&=\frac{\Gamma(\beta_{p-2})\Gamma(\beta_{p-1})}{\Gamma(\alpha_p)\Gamma(\beta_{p-2}+\beta_{p-1}-\alpha_p)}\,{}_{p-1}F_{p-2}\left[\begin{matrix}(\alpha_{p-1});\\(\beta_{p-3}),\ \beta_{p-2}+\beta_{p-1}-\alpha_p;\end{matrix}\; 1-\rho\right]\\&+\frac{(\beta_{p-2}-\alpha_p)(\beta_{p-1}-\alpha_p)\Gamma(\beta_{p-2})\Gamma(\beta_{p-1})}{\Gamma(\alpha_p)\Gamma(\beta_{p-2}+\beta_{p-1}-\alpha_p+1)}\\&\cdot F^{0:p-1;3}_{1:p-3;1}\left[\begin{matrix}\text{---}:&(\alpha_{p-1});&\beta_{p-2}-\alpha_p+1,\ \beta_{p-1}-\alpha_p+1,\ 1;\\\beta_{p-2}+\beta_{p-1}-\alpha_p+1;&(\beta_{p-3});&2;\end{matrix}\; 1-\rho,\ 1\right]\\&\qquad(p\geqq 3;\ \rho>0;\ \mathrm{Re}(\alpha_p)>0).\end{aligned}$$

It is not difficult to verify that the Kampé de Fériet series, occurring on the right-hand side of (7), converges when $\rho\to 0+$, provided that

$$(8)\qquad \mathrm{Re}\left(\sum_{j=1}^{p-1}\beta_j-\sum_{j=1}^{p}\alpha_j\right)>-1.$$

Obviously, this last condition (8) is satisfied if (for example) the hypergeometric series ${}_pF_{p-1}$, occurring on the left-hand side of (7), is zero-balanced.

Now we set $p=3$ in (7) and make use of the well-known result (cf. [4, p. 110, Equation 2.10(12)]; see also [11, p. 63, Equation (1)]):

(9)

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,{}_2F_1\left[\begin{matrix}a,\ b;\\a+b;\end{matrix}\; 1-\rho\right]=-[2\gamma+\psi(a)+\psi(b)+\log\rho]+o(1)\,(\rho\to 0+),$$

where, as usual, $\psi(z)=\Gamma'(z)/\Gamma(z)$, and γ denotes the Euler–Mascheroni constant. By virtue of (5), we thus obtain the formula:

(10)

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\beta_1)\Gamma(\beta_2)}\,{}_3F_2\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3;\\ \beta_1,\beta_2;\end{matrix}\;1-\rho\right]=-[2\gamma+\psi(\alpha_1)+\psi(\alpha_2)+\log\rho]$$

$$+\frac{(\beta_1-\alpha_3)(\beta_2-\alpha_3)}{\alpha_1\alpha_2}\,{}_4F_3\left[\begin{matrix}\beta_1-\alpha_3+1,\ \beta_2-\alpha_3+1,\ 1,1;\\ \alpha_1+1,\ \alpha_2+1,2;\end{matrix}\;1\right]$$

$$+o(1)(\rho\to 0+)$$

$$(\alpha_1+\alpha_2+\alpha_3=\beta_1+\beta_2;\ \mathrm{Re}(\alpha_3)>0),$$

which, as we indicated above, was considered by (among others) Ramanujan [9], Berndt [2], Saigo [12], Evans and Stanton [6], Evans [5], and Bühring [3].

In a similar manner, setting $p=4$ in (7) and applying the formula (10), we find that

(11)

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_4)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}\,{}_4F_3\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_4;\\ \beta_1,\beta_2,\beta_3;\end{matrix}\;1-\rho\right]=-[2\gamma+\psi(\alpha_1)+\psi(\alpha_2)+\log\rho]$$

$$+\frac{(\beta_1-\alpha_3)(\beta_2+\beta_3-\alpha_3-\alpha_4)}{\alpha_1\alpha_2}\,{}_4F_3\left[\begin{matrix}\beta_1-\alpha_3+1,\ \beta_2+\beta_3-\alpha_3-\alpha_4+1,\ 1,1;\\ \alpha_1+1,\ \alpha_2+1,2;\end{matrix}\;1\right]$$

$$+\frac{(\beta_2-\alpha_4)(\beta_3-\alpha_4)\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\beta_1)\Gamma(\beta_2+\beta_3-\alpha_4+1)}$$

$$\cdot F^{0:3;3}_{1:1;1}\left[\begin{matrix}\text{———}:&\alpha_1,\alpha_2,\alpha_3;&\beta_2-\alpha_4+1,\beta_3-\alpha_4+1,1;\\ \beta_2+\beta_3-\alpha_4+1:&\beta_1;&2;\end{matrix}\;1,1\right]$$

$$+o(1)(\rho\to 0+)$$

$$(\alpha_1+\alpha_2+\alpha_3+\alpha_4=\beta_1+\beta_2+\beta_3;\ \min\{\mathrm{Re}(\alpha_3),\mathrm{Re}(\alpha_4)\}>0).$$

Iterating this process, we are led from (7) to the general formula:

(12)

$$\frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}{\Gamma(\beta_1)\cdots\Gamma(\beta_{p-1})}\,{}_pF_{p-1}\left[\begin{matrix}(\alpha_p);\\(\beta_{p-1});\end{matrix}\;1-\rho\right] = -[2\gamma+\psi(\alpha_1)+\psi(\alpha_2)+\log\rho]$$

$$+\frac{\beta_1-\alpha_3}{\alpha_1\alpha_2}\left(\sum_{j=2}^{p-1}\beta_j-\sum_{j=3}^{p}\alpha_j\right){}_4F_3\left[\begin{matrix}\beta_1-\alpha_3+1,\ \sum_{j=2}^{p-1}\beta_j-\sum_{j=3}^{p}\alpha_j+1,\ 1,\ 1;\\ \alpha_1+1,\ \alpha_2+1,\ 2;\end{matrix}\;1\right]$$

$$+\sum_{k=3}^{p-1}\frac{(\beta_{k-1}-\alpha_{k+1})\left(\sum_{j=k}^{p-1}\beta_j-\sum_{j=k+1}^{p}\alpha_j\right)\Gamma(\alpha_1)\cdots\Gamma(\alpha_k)}{\Gamma(\beta_1)\cdots\Gamma(\beta_{k-2})\Gamma\left(\sum_{j=k-1}^{p-1}\beta_j-\sum_{j=k+1}^{p}\alpha_j+1\right)}$$

$$\cdot F^{0:\ \ k;3}_{1:k-2;1}\left[\begin{matrix}\text{———————}:\ (\alpha_k);\\ \sum_{j=k+1}^{p-1}\beta_j-\sum_{j=k-1}^{p}\alpha_j+1:\ (\beta_{k-2});\end{matrix}\right.$$

$$\left.\begin{matrix}\beta_{k-1}-\alpha_{k+1}+1,\ \sum_{j=k}^{p-1}\beta_j-\sum_{j=k+1}^{p}\alpha_j+1,\ 1;\\ 2;\end{matrix}\;1,1\right]$$

$$+\,o(1)(\rho\to 0+)$$

$$(p\geqq 3;\ \alpha_1+\cdots+\alpha_p=\beta_1+\cdots+\beta_{p-1};\ \mathrm{Re}(\alpha_j)>0,\ j=3,4,\ldots,p).$$

Since an empty sum is interpreted as zero, (12) yields the formulas (10) and (11) in its special cases when $p=3$ and $p=4$, respectively. Thus, in view of (7), (10), and (11), the general result (12) can be proven directly by induction on p.

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