

A NOTE ON A THEOREM OF MAY CONCERNING COMMUTATIVE GROUP ALGEBRAS

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ABSTRACT. Let G be a coproduct of p -primary abelian groups with each factor of cardinality not exceeding \aleph_1 , and let F be a perfect field of characteristic p . If $V(G)$ is the group of normalized units of the group algebra $F(G)$, it is shown that G is a direct factor of $V(G)$ and that the complementary factor is simply presented. This generalizes a theorem of W. May, who proved the result in the case when G itself has cardinality not exceeding \aleph_1 and length not exceeding ω_1 .

Throughout let F denote a perfect field of characteristic $p \neq 0$ and let G denote a p -primary abelian group. In a recent paper, W. May [M2] proved the following result, where $V(G)$ denotes the group of normalized units in the group algebra $F(G)$; hence $V(G) = \{\sum c_i g_i \in F(G) : \sum c_i = 1\}$.

Theorem 1 ([M2]). *If $|G| \leq \aleph_1$ and the length of G does not exceed ω_1 , then G is a direct factor of $V(G)$ and the complementary factor is simply presented.*

Recall that a group is simply presented if it can be presented by generators and relations that involve at most two generators.

The main purpose of this note is to show that the condition on the length of G in the above theorem can be omitted. In other words, the theorem is proved for a much more general class of groups (inasmuch as there exists an abundance of abelian p -groups of cardinality not exceeding \aleph_1 with arbitrary prescribed length $\lambda < \omega_2$). Actually, we extend the theorem to any abelian p -group G which is a coproduct of groups with the cardinality of each factor not exceeding \aleph_1 ; of course such a G can have arbitrarily large cardinality.

We refer to [M1] and [M3] as examples of the connection between the direct factor theorem and the isomorphism problem: When does $F(G) \cong F(G')$ imply that $G \cong G'$?

As is customary for group algebras, we use the multiplicative notation for G even though G is abelian. Consequently, for an ordinal σ , G^σ is the translation of the more familiar $p^\sigma G$; since there is only one relevant prime,

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there is no ambiguity in the abbreviated notation. Thus we write $|x| = \sigma$ if $x \in G^\sigma \setminus G^{\sigma+1}$ while $|x| = \infty$ means that $x \in G^\sigma$ for every σ . Unless otherwise specified, $|x|$ will be computed in the largest containing group. If it is not clear from the context, we write $|x|_G$ to denote the height of x computed in G . If $H \subseteq G$ we define the *height spectrum* of H in G to be the set of ordinals $\{|h|_G : h \in H\}$. The subgroup H of the abelian p -group G is *isotype* if $H \cap G^\sigma = H^\sigma$, and H is *nice* in G if $(G/H)^\sigma = G^\sigma H/H$, for all σ . If H is both isotype and nice in G , it is said to be *balanced*. It is a well-known and not difficult fact that G is a balanced subgroup of $V(G)$. Also, we mention that H is nice in G if and only if each coset of H in G contains an element with maximal height.

Since simply presented groups are always balanced projective, all that needs to be proved to establish the preceding theorem is that $V(G)/G$ is simply presented. Under the stated hypothesis, May [M2] was able to show that $V(G)/G$ is simply presented by using the Axiom 3 characterization of simply presented groups [F]; that is, May demonstrated that $V(G)/G$ has “sufficiently many” nice subgroups. This approach also captures the spirit of the present note. We identify and focus on certain nice subgroups of $V(G)$. In some cases, however, we state the preliminary results in the more general setting of arbitrary abelian p -groups. An example of this is our first lemma. Before stating this lemma, we need one more definition. Two subgroups A and B of the abelian p -group G are called *compatible* and we write $A \parallel B$ if for each pair $(a, b) \in A \times B$ there exists $c \in A \cap B$ such that $|ac| \geq |ab|$. This notion was introduced in [H] and has played a significant role in a number of subsequent papers.

Lemma 1. *Let N be a subgroup of the p -group G . If each element of G is contained in some subgroup M with finite height spectrum in G for which $N \parallel M$, then N is nice in G .*

Proof. For an arbitrary but fixed element $g \in G$, consider the set $S = \{|gx| : x \in N\}$. By hypothesis, there exists a subgroup M of G containing g for which $\{|x| : x \in M\}$ is finite and $N \parallel M$. Therefore, if $x \in N$, there exists $c \in N \cap M$ such that $|gc| \geq |gx|$. Since $gc \in M$, there are only a finite number of possibilities for $|gc|$ and each of these belongs to S . Consequently, S must contain a largest element, and N is nice in G .

If $\alpha = \sum c_i g_i \in F(G)$, with each $c_i \in F$ and $g_i \in G$, recall that the *augmentation* of α , written $\text{aug}(\alpha)$, is defined by $\text{aug}(\alpha) = \sum c_i$. Thus, $V(G) = \{\alpha \in F(G) : \text{aug}(\alpha) = 1\}$. In order to apply Lemma 1 and similar results to the group $V(G)$, we need to know that certain subgroups of $V(G)$ are compatible.

Lemma 2. *Let A and B be arbitrary subgroups of G , then*

- (i) $G \parallel V(A)$,
- (ii) $GV(A) \parallel V(B)$,
- (iii) $GV(A) \parallel GV(B)$.

Proof. Recall that if $\alpha = \sum c_i g_i \in V(G)$, where $c_i \neq 0$ and the g_i 's are distinct elements of G , then $|\alpha| = \min\{|g_i|\}$. To verify part (i), simply take $\alpha = \sum c_i a_i \in V(A)$ with the a_i 's distinct elements of A and $c_i \neq 0$ in the field F and consider $\alpha a_1^{-1} = c_1 + \sum_{i>1} c_i a_i a_1^{-1}$. Observe that the height of αa_1^{-1} cannot be increased by multiplying by any group element $g \in G$, for $|\alpha a_1^{-1} g| > |\alpha a_1^{-1}|$ implies on the one hand that $|g| = |\alpha a_1^{-1}|$, while on the other hand it implies that $|g| = |c_1 g| \geq |\alpha a_1^{-1} g| > |\alpha a_1^{-1}|$. Clearly $a_1^{-1} \in G \cap V(A)$ and we have observed that $|\alpha a_1^{-1}| \geq |\alpha a_1^{-1}(a_1 g)| = |\alpha g|$ for any $g \in G$. Therefore $G \parallel V(A)$. To prove that (ii) holds, let $\alpha = \sum c_i b_i \in V(B)$. By collecting the b_i 's that are equal mod A , we can write $\alpha = E_1 b_1 + E_2 b_2 + \dots + E_n b_n$, where $E_i \in F(A \cap B)$ and $b_1, b_2, \dots, b_n \in B$ are distinct mod A . Since the augmentation of α is 1, we may assume without loss of generality that $\text{aug}(E_1) = c \neq 0$. Set $\beta = c^{-1} E_1 b_1$ and observe that $\beta \in GV(A) \cap V(B)$. Moreover, $\alpha \beta^{-1} = c + E_2' b_2 b_1^{-1} + \dots + E_n' b_n b_1^{-1}$, where $E_i' = E_i (c^{-1} E_1)^{-1} \in F(A \cap B)$. From this equation it is easily seen that $|\alpha \beta^{-1}| \geq |\alpha \beta^{-1} \gamma|$ for any $\gamma \in GV(A)$. The key here, of course, is that $1, b_2 b_1^{-1}, \dots, b_n b_1^{-1}$ are distinct mod A . Hence $|\alpha \beta^{-1} \gamma| > |\alpha \beta^{-1}|$ leads to the contradiction

$$|\alpha \beta^{-1} \gamma| \leq |c \gamma| = |\gamma| = |\alpha \beta^{-1}| < |\alpha \beta^{-1} \gamma|.$$

This completes the proof of (ii), and an obvious modification of the preceding yields (iii).

Corollary 1 ([M2]). *If A is a subgroup of the p -group G , then $GV(A)$ is nice in $V(G)$.*

Proof. Each element α of $V(G)$ is contained in $V(B)$ for some finite subgroup B of G . Clearly $V(B)$ has finite height spectrum. By Lemma 2, $GV(A) \parallel V(B)$. Therefore, $GV(A)$ is nice as a consequence of Lemma 1.

The following is a generalization (perhaps only a slight one) of the preceding corollary.

Corollary 2. *Let N be a subgroup of $V(G)$ with $G \subseteq N \subseteq V(G)$. If $N \parallel GV(B)$ for every finite subgroup B of G , then N is nice in $V(G)$.*

Proof. Under the hypothesis, we claim that $N \parallel V(B)$ for every finite subgroup B of G . Choose $\alpha \in N$ and $\beta \in V(B)$, and let $|\alpha \beta| = \lambda$. Since $N \parallel GV(B)$, there exists $g \in G$ and $\beta' \in V(B)$ such that $g \beta' \in N$ and $|\alpha g \beta'| \geq \lambda$. Note that $\beta' \in N$ since $g \in G \subseteq N$ and that $|g \beta' \beta^{-1}| \geq \lambda$. Using the fact that $G \parallel V(B)$ by virtue of Lemma 2, we can choose $b \in G \cap V(B) = B$ such that $|g b^{-1}| \geq \lambda$. From this and the inequality $|\alpha g \beta'| \geq \lambda$, we conclude that $|\alpha b \beta'| \geq \lambda$. Since $b \beta' \in N \cap V(B)$, we have shown that $N \parallel V(B)$ for every finite subgroup B of G . Lemma 1 implies that N is nice in $V(G)$.

The next proposition is not required for the proofs of the main theorems, but it establishes a class of nice subgroups of $V(G)$ larger than the $GV(A)$'s.

Definition. A subgroup S of $V(G)$ is called a *solid* subgroup if $\sum_{i \in I} c_i g_i \in S$, with distinct g_i 's in G , then $c^{-1} \sum_{j \in J} c_j g_j \in S$ whenever $J \subseteq I$ and $\text{aug}(\sum_{j \in J} c_j g_j) = c \neq 0$.

It is not difficult to show that there exist solid subgroups of $V(G)$ containing G which are not of the form $GV(A)$.

Proposition. *If $G \subseteq S \subseteq V(G)$ and S is a solid subgroup of $V(G)$, then S is nice in $V(G)$.*

Proof. With an argument very similar to that used in the proof of Lemma 2, we can show that $S \parallel GV(B)$ for every subgroup B of G . Indeed, if $\alpha \in S$, we can write $\alpha = E_1 g_1 + E_2 g_2 + \cdots + E_n g_n$, where $E_i \in F(B)$ and the g_i 's are distinct mod B . Without loss of generality, $\text{aug}(E_1) = c \neq 0$. Observe that $c^{-1} E_1 g_1 \in GV(B) \cap S$ by virtue of S being solid in $V(G)$. Let $\beta = c^{-1} E_1 g_1$ and observe that $\alpha \beta^{-1} = c + E'_2 g_2 g_1^{-1} + \cdots + E'_n g_n g_1^{-1}$, where $E'_i \in F(B)$. It should be clear now that $|\alpha \beta^{-1}| \geq |\alpha \beta^{-1} \gamma|$ whenever $\gamma \in GV(B)$, for the argument is essentially the same as in the case of Lemma 2. Thus, for every $\delta \in GV(B)$, $|\alpha \delta| = |\alpha \beta^{-1} (\beta \delta)| \leq |\alpha \beta^{-1}|$ and we have shown that $S \parallel GV(B)$. That S is nice in $V(G)$ now follows from Corollary 2.

Only one more preliminary is needed for the proof of our first theorem, and we state it in the generality of arbitrary abelian p -groups.

Lemma 3. *Suppose that A and B are compatible subgroups of the p -group G , where A is nice in G and B has finite height spectrum in G . If C is any subgroup of B , then AC is nice in G .*

Proof. Let $c \in C$. Since A is nice in G , $|cA|_{G/A} = |ca|_G$ for some $a \in A$. But $A \parallel B$ implies that $|cd|_G \geq |ca|_G$ for some $d \in A \cap B$. Therefore, $|cA|_{G/A} = |cd|_G$ since $d \in A$ and heights cannot decrease under a homomorphic image. Since $cd \in B$ and B has a finite height spectrum in G , we have demonstrated that CA/A has a finite height spectrum in G/A . Therefore, CA/A must be a nice subgroup of G/A . And since A is nice in G , this implies that AC is nice in G .

Recall that F is a perfect field of characteristic $p \neq 0$ and G is a p -primary abelian group.

Theorem 2. *If $|G| \leq \aleph_1$, then G is a direct factor of $V(G)$ and $V(G)/G$ is simply presented.*

Proof. Let $\{A_\alpha : \alpha < \omega_1\}$ be a filtration of G ; that is, we let

$$1 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\alpha \subseteq A_{\alpha+1} \subseteq \cdots \quad (\alpha < \omega_1)$$

be a smooth chain (smooth means $A_\beta = \bigcup_{\alpha < \beta} A_\alpha$ when β is a limit) of countable subgroups of $G = \bigcup_{\alpha < \omega_1} A_\alpha$. For each $\alpha < \omega_1$, let $\{B_{\alpha,n} : n < \omega_0\}$ be a filtration of A_α . In other words,

$$1 = B_{\alpha,0} \subseteq B_{\alpha,1} \subseteq \cdots \subseteq B_{\alpha,n} \subseteq \cdots \quad (n < \omega_0)$$

is a chain of finite subgroups of $A_\alpha = \bigcup_{n < \omega_0} B_{\alpha,n}$. From Corollary 1, we know

that the $GV(A_\alpha)$'s form a chain of nice subgroups of $V(G)$. According to Lemma 3, $\{GV(A_\alpha)V(B_{\alpha+1,n}): n < \omega_0\}$ is a chain of nice subgroups of $V(G)$ going from $GV(A_\alpha)$ to $GV(A_{\alpha+1})$. Moreover, Lemma 3 also implies that if C is any subgroup between $V(B_{\alpha+1,n})$ and $V(B_{\alpha+1,n+1})$, then $GV(A_\alpha)C$ is nice in $V(G)$. Therefore, there is a smooth chain of nice subgroups of $V(G)$ between G and $V(G)$, where the quotient of adjacent members of the chain is simple of order p . Since G is balanced in $V(G)$, this means that $V(G)/G$ is simply presented and that G is a direct factor of $V(G)$.

We separate part of the proof of our final theorem as the following lemma.

Lemma 4. *If H is a direct factor of G , then $GV(H)/G$ is a direct factor of $V(G)/G$. Consequently, $GV(H)$ is a balanced subgroup of $V(G)$.*

Proof. The projection of G onto H induces a split epimorphism $V(G) \rightarrow V(H)$. Thus, $V(G) = V(H) \times K$ for some subgroup K of $V(G)$. From this it easily follows that $V(G)/G = GV(H)/G \times KG/G$. Finally, $GV(H)$ is balanced in $V(G)$, since G is balanced in $V(G)$ and $GV(H)/G$ is balanced $V(G)/G$.

Theorem 3. *Suppose $G = \coprod_{i \in I} G_i$ is a coproduct of p -groups G_i . If $|G_i| \leq \aleph_1$ for all $i \in I$, then G is a direct factor of $V(G)$ and $V(G)/G$ is simply presented.*

Proof. We show that $V(G)/G$ is simply presented by induction on $|G|$. If $|G| \leq \aleph_1$, the result follows from Theorem 2. Therefore, we may assume that $|G| > \aleph_1$. In this case, $|I| = |G|$. Let μ be the smallest ordinal of cardinality $|G|$; so we can take $I = \mu$ and $G = \coprod_{i < \mu} G_i$. For each $\alpha < \mu$, set $H_\alpha = \coprod_{i < \alpha} G_i$.

Since H_α is a direct factor of $H_{\alpha+1}$, Lemma 4 implies that $H_{\alpha+1}V(H_\alpha)/H_{\alpha+1}$ is a direct factor of $V(H_{\alpha+1})/H_{\alpha+1}$. By the induction hypothesis, the latter group is simply presented and hence so is

$$V(H_{\alpha+1})/H_{\alpha+1}V(H_\alpha) \cong (V(H_{\alpha+1})/H_{\alpha+1})/(H_{\alpha+1}V(H_\alpha)/H_{\alpha+1}).$$

Also by Lemma 4, $H_{\alpha+1}V(H_\alpha)$ is a balanced subgroup of $V(H_{\alpha+1})$. Therefore, $V(H_{\alpha+1}) = H_{\alpha+1}V(H_\alpha) \times T_\alpha$ for some simply presented subgroup T_α . Due to the smoothness of the sequence of H_α 's, it now follows easily that $V(G) = G \times \coprod_{\alpha < \mu} T_\alpha$, thus proving the theorem.

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