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NONEXISTENCE OF POSITIVELY EXPANSIVE MAPS ON COMPACT CONNECTED MANIFOLDS WITH BOUNDARY

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Dedicated to Professor Ryosuke Nakagawa on his sixtieth birthday

ABSTRACT. In this note we prove that no compact connected manifold with boundary admits a positively expansive map.

Let (X, d) be a compact metric space and $f\colon X \to X$ be a continuous map (not always surjective). We say that f is *positively expansive* if there is a constant $c > 0$ such that if $x, y \in X$ and $x \neq y$ then $d(f^i(x), f^i(y)) > c$ for some $i \geq 0$ (c is called an *expansive constant* for f). The notion of positive expansiveness is independent of the metrics compatible with original topology, and preserved under topological conjugacy. In D. W. Curtis and S. Miklos [3] they proved that no positively expansive map of X onto X can be a homeomorphism if X is nontrivial and connected. In fact, whenever X admits a positively expansive homeomorphism f, X is finite (cf. [1, 5]). This is easily checked as follows. By a result of W. L. Reddy [10], f is expanding with respect to some metric ρ for X, i.e. there exist constants $\delta > 0$ and $0 < \lambda < 1$ such that $\rho(x, y) < \delta$ implies $\rho(f^{-1}(x), f^{-1}(y)) \leq \lambda\rho(x, y)$. Thus $\Phi^- = \{f^{-i}\colon i \geq 0\}$ is uniformly equicontinuous. By a metric D defined by $D(f, g) = \max\{\rho(f(x), g(x))\colon x \in X\}$, Φ^- is totally bounded. Let $\Phi^+ = \{f^i\colon i \geq 0\}$ and define a map $G\colon \Phi^- \to \Phi^+$ by $G(f^{-i}) = f^i$ for $i \geq 0$. Then G is D-isometric. Therefore Φ^+ is totally bounded. Since X is compact, Φ^+ is uniformly equicontinuous and so there is $\varepsilon > 0$ such that $\rho(x, y) < \varepsilon$ implies $\rho(f^i(x), f^i(y)) < c$ for all $i \geq 0$ (c is an expansive constant for f). This shows $x = y$ and therefore x is an isolated point.

The study of positively expansive maps is an interesting subject in topological dynamics. In [3] the following is posed: "Characterize all compact connected manifolds which admit positively expansive maps." Our aim is to give an answer for this problem.

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Theorem 1. *No compact connected manifold with boundary admits a positively expansive map.*

M. Shub [11, 12], J. Franks [4], M. W. Hirsch [9], and M. Gromov [6] studied the problem of characterization of expanding differentiable maps on closed smooth manifolds. In [6] M. Gromov proved finally that an expanding differentiable map of an arbitrary closed smooth manifold is topologically conjugate to an expanding infra-nil-endomorphism. On the other hand, E. M. Coven and W. L. Reddy [2] studied positively expansive maps of closed topological manifolds and showed that such maps are expanding with respect to certain metrics. Recently the author [8] generalized a result of M. Gromov [6] as follows: a positively expansive map of an arbitrary closed topological manifold is topologically conjugate to an expanding infra-nil-endomorphism. Combining this and Theorem 1, we can conclude the following

Theorem 2. *Every compact connected manifold which admits a positively expansive map is homeomorphic to an infra-nil-manifold, and such a map of an infra-nil-manifold is topologically conjugate to an expanding infra-nil-endomorphism.*

Theorem 1 will be obtained by using the following.

Lemma. *Let X be a compact connected locally connected metric space and $f: X \to X$ be a positively expansive map. If a closed proper subset K of X satisfies the following conditions:*

(1) $f(X\backslash K) \subset X\backslash K$,
(2) $f_{|X\backslash K}: X\backslash K \to X\backslash K$ *is an open map,*

then $K = \phi$ (compare with Theorem 3 of [3]).

To use the lemma for the proof of Theorem 1, let M be a compact connected manifold and ∂M denote the boundary of M. Suppose that M admits a positively expansive map f. From the definition it follows that f is locally injective. Combining this fact and Brouwer's theorem on invariance of domain, we have that $f(M\backslash\partial M) \subset M\backslash\partial M$ and $f_{|M\backslash\partial M}: M\backslash\partial M \to M\backslash\partial M$ is an open map. Use the lemma here, then we have $\partial M = \phi$.

We must prove the lemma to obtain the conclusion. Let $f: X \to X$ be as in the lemma. Then there exist a metric ρ for X and constants $\delta > 0$ and $\lambda > 1$ such that if $\rho(x, y) < \delta$ then $\rho(f(x), f(y)) \geq \lambda\rho(x, y)$. This is proved in the same way as in [10] (notice that f is not always surjective). For $\epsilon > 0$ and $x \in X$, let $U_\epsilon(x) = \{y \in X: \rho(x, y) < \epsilon\}$ and denote by $C_\epsilon(x)$ the connected component of x in $U_\epsilon(x)$. Obviously $C_\epsilon(x)$ is open in X. Since X is locally arcwise connected (Theorem 6.29 of [7]), we have that $C_\epsilon(x)$ is arcwise connected.

We first check the following claim: Let $0 < \epsilon < \delta/2$ and $x \in X\backslash K$. If $C_\epsilon(x) \subset X\backslash K$, then $f(C_\epsilon(x)) \supset C_{\lambda\epsilon}(f(x))$.

Assume $y \in C_{\lambda\epsilon}(f(x))\backslash f(C_\epsilon(x)) \neq \phi$. Since $C_{\lambda\epsilon}(f(x))$ is arcwise connected, there exists an arc $\omega: [0, 1] \to C_{\lambda\epsilon}(f(x))$ such that $\omega(0) = f(x)$ and $\omega(1) = y$.

Since $C_\epsilon(x) \subset X\backslash K$, $f(C_\epsilon(x))$ is open in X by (2). Obviously $\omega(0) = f(x) \in f(C_\epsilon(x))$. Hence we can take $0 < t_0 < 1$ such that $\omega([0, t_0)) \subset f(C_\epsilon(x))$ and $\omega(t_0) \notin f(C_\epsilon(x))$. Then $\omega([0, t_0]) \subset \overline{f(C_\epsilon(x))}$ where $\overline{f(C_\epsilon(x))}$ denotes the closure of $f(C_\epsilon(x))$ in X.

Obviously $f(\overline{C_\epsilon(x)}) = \overline{f(C_\epsilon(x))}$. Since $\overline{C_\epsilon(x)} \subset U_{\delta/2}(x)$, by the choice of δ we have that $f|_{\overline{C_\epsilon(x)}}: \overline{C_\epsilon(x)} \to \overline{f(C_\epsilon(x))}$ is injective, and so $f|_{\overline{C_\epsilon(x)}}: \overline{C_\epsilon(x)} \to \overline{f(C_\epsilon(x))}$ is a homeomorphism. Since $\omega([0, t_0]) \subset \overline{f(C_\epsilon(x))}$, we can find an arc $\overline{\omega}: [0, t_0] \to \overline{C_\epsilon(x)}$ such that $f \circ \overline{\omega} = \omega$. Note that $\omega([0, t_0)) \subset f(C_\epsilon(x))$. Then we have that $\overline{\omega}([0, t_0)) \subset C_\epsilon(x)$. Since $\omega(t_0) \notin f(C_\epsilon(x))$, obviously $\omega(t_0) \notin C_\epsilon(x)$.

On the other hand, since $\omega(t_0) \in C_{\lambda\epsilon}(f(x))$, we have

$$\lambda\rho(\overline{\omega}(t_0), x) \leq \rho(\omega(t_0), f(x)) < \lambda\epsilon,$$

and so $\overline{\omega}(t_0) \in U_\epsilon(x)$. Combining this and the fact that $x \in \overline{\omega}([0, t_0)) \subset C_\epsilon(x) \subset U_\epsilon(x)$, we have that $\overline{\omega}(t_0) \in C_\epsilon(x)$, thus a contradiction.

We proceed to the proof of the lemma. For $\epsilon > 0$ let

$$X(\epsilon) = \{x \in X\backslash K : C_\epsilon(x) \subset X\backslash K\}.$$

Since K is a closed proper subset of X, there exist $0 < \epsilon_0 < \delta/2$ such that $X(\epsilon_0) \neq \phi$. Assume that $K \neq \phi$. Obviously $X(\epsilon_0) \subsetneqq X$. Let $x \in X(\epsilon_0)$. Then $C_{\epsilon_0}(x) \subset X\backslash K$. From the above claim $f(C_{\epsilon_0}(x)) \supset C_{\lambda\epsilon_0}(f(x))$. Hence $C_{\lambda\epsilon_0}(f(x)) \subset X\backslash K$ by (1) and so $f(x) \in X(\lambda\epsilon_0)$. Therefore $f(X(\epsilon_0)) \subset X(\lambda\epsilon_0)$.

It is easily checked that $X(\lambda\epsilon_0) \subset X(\mu\epsilon_0) \subset X(\epsilon_0)$ for $1 < \mu < \lambda$. We show that $\overline{X(\lambda\epsilon_0)} \subset X(\mu\epsilon_0)$. To do this, let $\{x_i\}_{i\geq 0}$ be a sequence of $X(\lambda\epsilon_0)$ and let $x_i \to x \in X$ as $i \to \infty$. Obviously $U_{\mu\epsilon_0}(x) \subset U_{\lambda\epsilon_0}(x_i)$ for sufficiently large i. Since $C_{\mu\epsilon_0}(x)$ is oepn in X, we may assume that $x_i \in C_{\mu\epsilon_0}(x)$. Then $C_{\mu\epsilon_0}(x) \subset C_{\lambda\epsilon_0}(x_i)$, which implies $x \in X(\mu\epsilon_0)$. Therefore $\overline{X(\lambda\epsilon_0)} \subset X(\mu\epsilon_0)$.

Consequently we have

$$f(\overline{X(\epsilon_0)}) = \overline{f(X(\epsilon_0))} \subset \overline{X(\lambda\epsilon_0)} \subset X(\mu\epsilon_0) \subset X(\epsilon_0).$$

Hence $Y = \bigcap_{i\geq 0} f^i(\overline{X(\epsilon_0)})$ is a nonempty closed set and $f(Y) = Y$. Since $Y \subset X(\epsilon_0) \subsetneqq X$ and X is connected, obviously Y is not open in X.

For $A \subset X$ and $\alpha > 0$ let $N_\alpha(A) = \bigcup_{a\in A} C_\alpha(a)$. Then we have $N_{(\mu-1)\epsilon_0}(Y) \subset X(\epsilon_0)$. This is checked as follows. Let $z \in Y$ and $x \in C_{(\mu-1)\epsilon_0}(z)$. Then $C_{\epsilon_0}(x) \subset C_{\mu\epsilon_0}(z)$. Since $z \in Y \subset X(\mu\epsilon_0)$, $C_{\epsilon_0}(x) \subset C_{\mu\epsilon_0}(z) \subset X\backslash K$ and so $x \in X(\epsilon_0)$.

By the above result $Y \subset N_{(\mu-1)\epsilon_0}(Y) \subset \overline{X(\epsilon_0)}$. Since $f(Y) = Y$, we have

$$Y \subset \bigcap_{i\geq 0} f^i(N_{(\mu-1)\epsilon_0}(Y)) \subset \bigcap_{i\geq 0} f^i(\overline{X(\epsilon_0)}) = Y,$$

and hence $Y = \bigcap_{i\geq 0} f^i(N_{(\mu-1)\epsilon_0}(Y))$.

On the other hand, let $z \in Y$. Then $C_{(\mu-1)\epsilon_0}(z) \subset C_{\mu\epsilon_0}(z) \subset X\backslash K$. From the claim we have $f(C_{(\mu-1)\epsilon_0}(z)) \supset C_{\lambda(\mu-1)\epsilon_0}(f(z))$. Hence $f(N_{(\mu-1)\epsilon_0}(Y)) \supset N_{\lambda(\mu-1)\epsilon_0}(Y) \supset N_{(\mu-1)\epsilon_0}(Y)$ (since $f(Y) = Y$.) Therefore $Y = N_{(\mu-1)\epsilon_0}(Y)$. This implies that Y is open in X. We arrived at a contradiction.

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