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ON HANKEL TRANSFORM

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ABSTRACT. In this paper, using some results of the author on Hankel transform in the Schwartz and Gel'fand–Shilov spaces, we characterize the integral operators of Hankel type which are isomorphisms between the spaces H_μ of Zemanian. As a particular case, we obtain the classical Zemanian results on Hankel transform, some results of Mendez, and improve some results of Lee. Finally, we use these results to characterize the functions f of the Schwartz space which satisfy $\int_0^\infty t^{\alpha+n} f(t)\,dt = 0$ for all $n \geq 0$ and $\alpha > -1$.

INTRODUCTION

In [9], Zemanian defined the spaces H_μ $(\mu \geq -\frac{1}{2})$ as

$$(1) \qquad H_\mu = \{f \in \mathscr{C}^\infty((0, +\infty)) \colon \forall k, n \geq 0 \ |t^k (t^{-1}D)^n t^{-1/2-\mu} f(t)| < C_{k,n}\}$$

and he proved that the Hankel transform:

$$(2) \qquad H_\mu(f)(x) = \int_0^\infty f(t)\sqrt{xt}\,J_\mu(xt)\,dt$$

is an automorphism of the space H_μ.

In [5], Lee generalized these spaces in the following way (similar to the Gel'fand–Shilov spaces), given α, $\beta \geq 0$, A, $B > 0$, and $\mu \geq -\frac{1}{2}$, the space $H^{\beta,B}_{\mu,\alpha,A}$ is defined by

$$(3) \quad H^{\beta,B}_{\mu,\alpha,A} = \{f \in \mathscr{C}^\infty((0, +\infty)) \colon \exists C, \delta, \rho > 0 \text{ such that } \forall k, n \geq 0$$
$$|t^k (t^{-1}D)^n t^{-1/2-\mu} f(t)| < C(A+\delta)^n (B+\rho)^k k^{\alpha k} n^{\beta n}\}$$

and

$$(4) \qquad H^\beta_{\mu,\alpha} = \bigcup_{A,B\geq 1} H^{\beta,B}_{\mu,\alpha,A}.$$

He studied the nontriviality of these spaces and the Hankel operator (2).

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In this paper, we obtain the spaces H_μ and $H^\alpha_{\mu,\beta}$ through the spaces S^+ and $S^{+\beta}_\alpha$ (see [2, 3]):

$$S^+ = \{f \in C^\infty((0, +\infty)) : \forall k, n \geq 0 |t^k f^{(n)}(t)| < C_{k,n}\} \tag{5}$$

(6)
$$S^{+\beta}_\alpha = \{f \in \mathscr{C}^\infty(0, +\infty) : \exists C, A, B > 0\ \forall k, n \geq 0 |t^k f^{(n)}(t)| < CA^n B^k k^{\alpha k} n^{\beta n}\}$$

Using some results of [2, 3], we characterize the operators defined in a similar form as (2) which are isomorphism between the spaces H_α and H_β. As particular cases, we obtain the classical results of Zemanian in [9], those of Mendez in [7] and we improve those of Lee in [5, 6].

Finally, we use our theorems to characterize the functions f of the space S^+ which satisfy $\int_0^\infty t^{\alpha+n} f(t)\,dt = 0$ for all $n \geq 0$ and $\alpha > -1$ (this should be compared with Theorem 4.1 and 4.4 of [4]). We also solve the problem which we had proposed in [4].

Preliminaries

We consider the integral operators $\mathscr{H}_\alpha$:

$$\mathscr{H}_\alpha(f)(x) = \frac{1}{2}\int_0^\infty f(t)(tx)^{-\alpha/2} t^\alpha J_\alpha(\sqrt{xt})\,dt. \tag{7}$$

Now, we state a theorem for the space S^+, which was proved in [2, Theorem 6.2] for the space S^{+0}_1. The proof is based on [3, Theorem 2.5] and is analogous to that given in [2]. For $\alpha \geq 0$, another proof of this theorem can be found in [8].

Theorem 1. *For $\alpha > -1$, the integral operator defined by (7) is an isomorphism of S^+ onto itself and $\mathscr{H}_\alpha^2 = \mathrm{Id}$.*

Now, we shall prove that this Theorem is the best possible in the following sense.

Theorem 2. *Let $\alpha > -1$ and H be the integral operator defined by $H(f)(x) = \int_0^\infty f(t) x^a t^b J_\alpha(\sqrt{xt})\,dt$. Then H is an isomorphism of S^+ onto itself if and only if $a = -\alpha/2$ and $b = \alpha/2$.*

Proof. We consider the following subspace of S^+ denoted by $S^+ \cap S$: a function $f \in S^+ \cap S$ if and only if $f \in S^+$ and $f^{(n)}(0) = 0$ for all $n \geq 0$.

Let us assume that H is an isomorphism of S^+ onto itself. Let $c \in \mathbf{R}$ such that $b + c = \alpha/2$. We suppose that $a + \alpha/2 \notin \mathbf{N}$. Let f, $g \in S^+ \cap S$ such that $f(t) = t^c g(t)$. Then

$$H(f)(x) = H(t^c g)(x) = x^a \int_0^\infty g(t) t^{b+c} J_\alpha(\sqrt{xt})\,dt = 2x^{a+\alpha/2}\mathscr{H}_\alpha(g)(x).$$

Since $a + \alpha/2 \notin \mathbf{N}$ and $H(f) \in S^+$, we deduce that $\mathscr{H}_a(g)(0) = 0$ for all $g \in S^+ \cap S$. Now, let g be a C^∞-function with support contained in [0,1] and

such that $\int_0^\infty t^\alpha g(t)\,dt \neq 0$. As

$$J_\alpha(x) = c_\alpha x^\alpha \sum_k a_k x^{2k} \tag{8}$$

we have

$$\mathscr{H}_\alpha g(x) = \frac{1}{2}\int_0^\infty g(t)(tx)^{-\alpha/2} t^\alpha J_\alpha(\sqrt{xt})\,dt = \frac{c_\alpha}{2}\int_0^\infty g(t)t^\alpha \sum_k a_k (xt)^k\,dt. \tag{9}$$

Now, $x^{-\alpha}J_\alpha(x)$ is bounded as x tends to 0 and from the asymptotic expansion of J_α, we see that $x^{1/2}J_\alpha(x)$ is bounded when x tends to ∞, so we get

$$|(xt)^{-\alpha/2}J_\alpha(\sqrt{xt})| \leq C \quad \text{if } 0 < x,\, t. \tag{10}$$

From (9) and (10), we deduce that $\mathscr{H}_\alpha g(0) = c_\alpha \int_0^\infty a_0 g(t) t^\alpha\,dt \neq 0$. As $g \in S^+ \cap S$, this shows that $a + \alpha/2 \in \mathbf{N}$.

Now, we suppose that $c \notin \mathbf{N}$, so that $t^c e^{-t/2} \notin S^+$. Since H is an isomorphism of S^+ onto itself, $H(t^c e^{-1/2}) \notin S^+$, but $H(t^c e^{-t/2}) = 2x^{a+\alpha/2}\mathscr{H}_\alpha(e^{-t/2})$, and as $a + a/2 \in \mathbf{N}$ it follows that $x^{a+\alpha/2}\mathscr{H}_\alpha(e^{-t/2}) \in S^+$. Hence $c \in \mathbf{N}$.

Let n, $m \in \mathbf{N}$ such that $b = \alpha/2 - n$ and $a = -\alpha/2 + m$. By (8), we can write

$$\begin{aligned} H(f)(x) &= x^{-\alpha/2+n}\int_0^\infty f(t)t^{-\alpha/2+m}J_\alpha(\sqrt{xt})\,dt \\ &= c_\alpha x^n \int_0^\infty f(t)t^{\alpha-m}\sum_k a_k (xt)^k\,dt. \end{aligned}$$

If $n > 0$, from (10) it follows that $H(f)(0) = 0$ for all $f \in S^+$, and so H would not be an isomorphism of S^+ onto itself. Therefore $n = 0$.

Now, if $m > 0$, as $t^{-m}e^{-t/2} \notin S^+$, it follows that $\mathscr{H}_\alpha(t^{-m}e^{-t/2}) \notin S^+$ but $\mathscr{H}_\alpha(t^{-m}e^{-t/2}) = \frac{1}{2}H(e^{-t/2}) \in S^+$. So $m = 0$ and the theorem is proved.

The Hankel transform. Now, we shall use the preliminary results to study the Hankel transform in the spaces H_α.

Let f, g be two functions such that $f(t) = g(t^2)$. Formally, it is clear that if $x = t^2$, then $(\frac{1}{t}\frac{d}{dt})^n(f(t)) = 2^n(\frac{d}{dx})^n(g(x))$. Using this formula, we obtain the following relation between the spaces defined in the introduction.

Lemma 3. *Let* $\mu \geq -\frac{1}{2}$, α, $\beta \geq 0$. *Then*

$$H_{-1/2} = \{g(t^2) \colon g \in S^+\}, \tag{11}$$

$$H_\mu = t^{1/2+\mu}\{g(t^2) \colon g \in S^+\}, \tag{12}$$

$$H^\beta_{\mu,\alpha} = t^{1/2+\mu}\{g(t^2) \colon g \in S^{+\beta}_{2\alpha}\}. \tag{13}$$

As these spaces are Frechet ((11), (12)) or LF spaces ((13)), using the Closed Graph theorem it is clear that $T_\mu \colon S^+(\text{resp. } S^{+\beta}_{2\alpha}) \to H_\mu(\text{resp.} H^\beta_{\mu,\alpha})$ defined by

$T_\mu(f)(t) = t^{1/2+\mu} f(t^2)$ is an isomorphism. Its inverse is given by $T_\mu^{-1}(f)(t) = t^{-1/4-\mu/2} f(\sqrt{t})$. From (13) and the Appendix in [2] Theorem 4 follows.

Theorem 4. *Let $\mu \geq -\frac{1}{2}$, α, $\beta \geq 0$. Then $H^\beta_{\mu,\alpha} = \{0\}$ if and only if*

(a) $\alpha \neq 0$, $\beta \neq 0$, *and* $\beta + 2\alpha < 1$, *or*
(b) $\alpha = 0$ *and* $\beta \leq 1$, *or*
(c) $\beta = 0$ *and* $\alpha < \frac{1}{2}$.

This theorem improves the results of Lee in [5], since he only proved that $H^0_{\mu,0} = \{0\}$ and $H^0_{\mu,1/2} \neq \{0\}$.

Using Lemma 3 and Theorem 1, we obtain the following commutative diagram:

$$\begin{array}{ccc} S^+ & \overset{\mathscr{H}_\mu}{\leftrightarrow} & S^+ \\ {\scriptstyle T_\alpha}\updownarrow & & \updownarrow{\scriptstyle T_\beta} \\ H_\alpha & \overset{L_{\beta,\mu,\alpha}}{\leftrightarrow} & H_\beta \end{array}$$

where

$$L_{\beta,\mu,\alpha} = T_\beta \circ \mathscr{H}_\mu \circ T_\alpha^{-1}$$

is an isomorphism of H_α onto H_β. Its inverse is $L^{-1}_{\beta,\mu,\alpha} = T_\alpha \circ \mathscr{H}_\mu \circ T_\beta^{-1}$. So, when $\alpha = \beta$, $L^2_{\beta,\mu,\alpha} = \mathrm{Id}$. It is easy to check that

$$L_{\beta,\mu,\alpha}(f)(x) = x^{\beta+1/2-\mu} \int_0^\infty f(u) u^{\mu-\alpha+1/2} J_\mu(ux)\, du$$

and

$$L^{-1}_{\beta,\mu,\alpha}(f)(x) = x^{\alpha+1/2-\mu} \int_0^\infty f(u) u^{\mu-\beta+1/2} J_\mu(ux)\, du.$$

Now, from Theorem 2, we deduce the following theorem.

Theorem 5. *Let $\alpha > -\frac{1}{2}$ and H be the integral operator defined by $H(f)(x) = \int_0^\infty f(t) x^a t^b J_\mu(xt)\, dt$. Then H is an isomorphism of H_α onto H_β if and only if $a = \beta + \frac{1}{2} - \mu$ and $b = \mu - \alpha + \frac{1}{2}$. Moreover, $H^2 = \mathrm{Id}$ if and only if $\alpha = \beta$.*

Choosing α and β in a suitable way, we can obtain some classical results. For example:

(I) Taking $\alpha = \beta = \mu$ we obtain the classical result of Zemanian in [9], i.e.

$$L_{\mu,\mu,\mu} = H_\mu,$$

where H_μ is the Hankel transform defined by (2). So, H_μ is an isomorphism of H_μ onto H_μ, and $H_\mu^2 = \mathrm{Id}$.

(II) Taking $\alpha = \beta = -\frac{1}{2} + \mu$ and $\alpha = \beta = \frac{1}{2} + \mu$, we obtain Theorems 3 and 4 of Mendez in [7].

(III) Now, we'll improve the results of Lee in [6]. Indeed, we consider the spaces F_μ defined by Lee in [6]

$$F_\mu = \{f \in \mathscr{C}^\infty((0, +\infty)) \colon \exists C_{k,n} > 0 \text{ such that } |x^n (D^2 + (2\mu+1)(x^{-1}D))^k f(x)| < C_{k,n}\}.$$

By Remark 2 in [6], it is clear that $F_\mu = H_{-1/2}$ for all $\mu \geq -\frac{1}{2}$. So, if we take $\alpha = \beta = -\frac{1}{2}$ in the previous diagram, we have:

$$L_{-1/2,\mu,-1/2} = h_\mu(f)(x) = \int_0^\infty f(t)t^{2\mu+1}(tx)^{-\mu}J_\mu(tx)\,dt \tag{14}$$

and so, h_μ is an isomorphism of $H_{-1/2}$ onto itself, i.e. h_μ is an isomorphism of F_μ onto itself and $h_\mu^2 = \mathrm{Id}$. Lee had proved that h_μ is a continuous linear mapping from the space F_μ into the spaces G_μ, where $F_\mu \subset G_\mu$. Moreover, as the operator (7) is defined for $\alpha > -1$, the operator h_μ defined by (11), is an isomorphism for $\mu > -1$. This gives a partial answer to problem 1 in [6]. So, we have proved the following theorem:

Theorem 6. *Let $\mu > -1$, then the operator h_μ defined by* (14) *is an isomorphism of F_μ onto itself and $h_\mu^2 = \mathrm{Id}$.*

Dualizing, we can extend the operator h_μ to the distributional space F'_μ as follows: if $u \in F'_\mu$ then $h_\mu(u)$ is defined by

$$\langle h_\mu(u), f\rangle = \langle u, h_\mu(f)\rangle \quad \text{for all } f \in F_\mu. \tag{15}$$

The following Theorem follows from Theorem 6.

Theorem 7. *Let $\mu > -1$, then the operator h_μ defined by* (15) *is an isomorphism of F'_μ onto itself and $h_\mu^2 = \mathrm{Id}$.*

In a similar way, since the operator (7) is an automorphism of the space S_1^{+0}, from the diagram, (13) and taking $\alpha = \beta = \mu$, we obtain that the operator H_μ defined by (2) is an isomorphism of $H^0_{\mu,1/2}$ onto itself. This improves the results of Lee in [6].

Appendix

In this Appendix, we shall use Theorem 1 to give a characterization similar to [4, Theorem 4.1] of the functions in S^+ which satisfy $\int_0^\infty f(t)t^{\alpha+n}\,dt = 0$ for all $n \geq 0$ and $\alpha > -1$. We need the following results:

Let $\alpha > -1$, and $\mathscr{L}_n^\alpha(t) = (n!/(\Gamma(n+\alpha+1))^{1/2}L_n^\alpha(t)t^{\alpha/2}e^{-t/2}$, where $L_n^\alpha(t)$ are the generalized Laguerre polynomials. It is well known that $(\mathscr{L}_n^\alpha(t))_n$ is an orthonormal system in $L^2([0,\infty))$. If $\psi \in S^+$ we define the α Fourier-Laguerre coefficients of ψ as $a_n = \int_0^\infty t^{\alpha/2}\psi(t)\mathscr{L}_n^\alpha(t)\,dt$. From [3, Theorem 2.5] it follows that

Theorem A. *The mapping $\mathscr{L}: S^+ \to s$ defined by*

$$\mathscr{L}(\phi) = (a_n)_n = \left(\int_0^{+\infty} t^{\alpha/2}\phi(t)\mathscr{L}_n^\alpha(t)\,dt\right)n$$

is an isomorphism of the space S^+ onto s. (We write s for the space of rapidly decreasing sequences).

Now, we can complete Theorem 1. Indeed, as (see [2, 6.1])

$$\int_0^\infty t^{\alpha/2} L_n^\alpha(t) e^{-t/2} J_\alpha(\sqrt{xt})\, dt = 2(-1)^n x^{\alpha/2} L_n^\alpha(x) e^{-x/2},$$

it is easy to prove (see [2, proof of Theorem 6.2]) that the α Fourier-Laguerre coefficients of $\mathscr{H}_\alpha(f)$ are $((-1)^n a_n^{(\alpha)})_n$ where $(a_n^{(\alpha)})_n$ are the α Fourier-Laguerre coefficients of f.

Let $\phi \in L^1([0, +\infty))$, and consider the following analytic function $\tilde{\phi}$ in the lower half plane:

$$\tilde{\phi}(x) = \int_0^{+\infty} \phi(t) e^{-2\pi i x t}\, dt.$$

Using the bilinear transformation $W(z) = (-\frac{1}{2} + 2\pi i z)/(\frac{1}{2} + 2\pi i z)$, which transforms the lower half plane in the unit disc, we obtain the following analytic function ϕ in the unit disc:

$$\check{\phi}(w) = \tilde{\phi}(Z(w)) = \int_0^\infty \phi(t) e^{-\frac{1}{2}\left(\frac{1+w}{1-w} t\right)}\, dt.$$

In the following Theorem [4, Theorem 4.2], we relate the function $\check{\phi}$ with the Fourier-Laguerre coefficients of ϕ.

Theorem B. *Let* $\phi \in S^+$ *and* $a_n^\alpha = \int_0^\infty t^{\alpha/2} \phi(t) \mathscr{L}_n^\alpha(t)\, dt$. *Then* $(t^\alpha \phi)^\smile(w) = (1-w)^{\alpha+1} \sum_{n=0}^\infty a_n^\alpha (\Gamma(n+\alpha+1)/n!)^{1/2} w^n$.

Now, we can prove the following theorem (to compare with [4, Theorems 4.1 and 4.4])

Theorem 8. *Let* $\phi \in S^+$, $\alpha > -1$, *and* $a_n^\alpha = \int_0^\infty t^{\alpha/2} f(t) \mathscr{L}_n^\alpha(t)\, dt$. *The following conditions are equivalent*:

(a) $\int_0^\infty t^{\alpha+k} f(t)\, dt = 0$ *for all* $k \geq 0$.
(b) $\sum_n (-1)^n n^k \binom{n+\alpha}{n}^{1/2} a_n^\alpha = 0$ *for all* $k \geq 0$.
(c) *There exists a function* $\psi \in S^+ \cap S$ *such that* $\phi = \mathscr{H}_\alpha(\psi)$.

Proof. (a) $\Leftrightarrow$ (b) It is clear that $\int_0^\infty \phi(t) t^{\alpha+k}\, dt = 0$ for all $k \geq 0$ if and only if $\lim_{z\to 0} \widetilde{t^\alpha \phi}^{(k)}(z) = 0$ for all $k \geq 0$ and so if and only if $\lim_{w\to -1} (t^\alpha \phi)^{\smile(k)}(w) = 0$ for all $k \geq 0$ (because $\check{\phi}(w) = \tilde{\phi}(Z(w))$ and $d^n Z(w)/dw^n$ is bounded near $w = -1$). It is clear that $\lim_{w\to -1} (t^\alpha \phi)^{\smile(k)}(w) = 0$ for all $k \geq 0$ if and only if $\lim_{w\to -1} ((t^\alpha \phi)^\smile(w)/(1-w)^{\alpha+1})^{(k)} = 0$. We have $(t^\alpha \phi)^\smile(w) = (1-w)^{\alpha+1} \sum_n a_n^\alpha (\Gamma(n+\alpha+1)/n!)^{1/2} w^n$ and $(a_n^\alpha)_n \in s$, so

$$\frac{(t^\alpha \phi)^\smile(w)}{(1-w)^{\alpha+1}} = \sum_n a_n^\alpha \left(\frac{\Gamma(n+\alpha+1)}{n!}\right)^{1/2} w^n$$

and

$$\left(\frac{(t^\alpha\phi)^\smile(w)}{(1-w)^{\alpha+1}}\right)^{(k)}$$
$$= \sum_n n^k a_n^\alpha \left(\frac{\Gamma(n+\alpha+1)}{n!}\right)^{1/2} w^{n-k}$$
$$+ b_{k-1}^{(k)} \sum_n n^{k-1} a_n^\alpha \left(\frac{\Gamma(n+\alpha+1)}{n!}\right)^{1/2} w^{n-k}$$
$$+ \cdots + b_1^{(k)} \sum_n n a_n^\alpha \left(\frac{\Gamma(n+\alpha+1)}{n!}\right)^{1/2} w^{n-k}$$

for certain $b_{k-1}^{(k)}, \cdots, b_1^{(k)} \in \mathbf{N}$ and $k \geq 1$. Applying successively theorems of Tauber and Abel, we get $\int_0^\infty \phi(t)t^{\alpha+k}\,dt = 0$ if and only if

$$\sum_n (-1)^n n^k a_n^\alpha (\Gamma(n+\alpha+1)/n!)^{1/2} = 0$$

for all $k \geq 0$. But $\binom{n+\alpha}{n} = \Gamma(n+\alpha+1)/n!\Gamma(\alpha)$.

(b) $\Leftrightarrow$ (c) By the remark following Theorem A the Fourier-Laguerre coefficients of $\mathscr{H}_\alpha(\phi)$ are $((-1)^n a_n^\alpha)_n$. As by [2, Theorem 3.2]

$$\sum_n (-1)^n n^k a_n^\alpha \binom{n+\alpha}{n}^{1/2} = 0$$

if and only if $\mathscr{H}_\alpha(\phi) \in S^+ \cap S$. Since $\mathscr{H}_\alpha^2 = \mathrm{id}$, setting $\psi = \mathscr{H}_\alpha(\phi)$ we finish the proof.

Finally, we solve the problem which we had proposed in [4]. Indeed, we prove the following theorem.

Theorem 9. *Let $\phi \in S^+$, and $a_n = \int_0^\infty \phi(t)L_n(t)e^{-t/2}\,dt$. The following conditions are equivalent*:

(a) $\phi \in S^+ \cap S$ *and* $\int_0^\infty t^k\phi(t)\,dt = 0$ *for all* $k \geq 0$.

(b) $\sum_n (-1)^n n^k a_n = 0$ *and* $\sum_n n^k a_n = 0$ *for all* $k \geq 0$.

(c) *There exist two functions* ψ, $\varphi \in S^+ \cap S$ *such that* $\phi = \varphi * \mathscr{H}_0\psi$.

Proof. By [4], we only need to prove that (a) implies (c). Let

$$P_1^\infty = \Big\{\text{the functions } f\text{, analytics in the open unit disc such that } f \text{ and all its derivatives are bounded in the unit disc and } \lim_{w\to 1} f^{(k)}(w) = 0 \text{ for all } k \geq 0\Big\}.$$

For (a) implies (c) it is equivalent to prove that if $f \in P_1^\infty$ and $\lim_{w\to -1} f^{(k)}(w) = 0$ for all $k \geq 0$, then there exist two functions g, h such that $f = g \cdot h$,

$g \in P_1^\infty$ and h is a bounded and analytic function in the open unit disc, with all its derivatives bounded in the unit disc and satisfying that $\lim_{w \to -1} h^{(k)}(w) = 0$. Now, given a function $f \in P_1^\infty$ such that $\lim_{w \to -1} f^{(k)}(w) = 0$ for all $k \geq 0$ by [1, Theorem 3, p. 22], there exist two functions g, $h \in P_1^\infty$ such that $f = g \cdot h$ and h belongs to the closed ideal generated by f. So $\lim_{w \to 1} g^{(k)}(w) = 0$ and $\lim_{w \to -1} h^{(k)}(w) = 0$ for all $k \geq 0$. And the theorem is proved.

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