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THE IRRATIONALS ARE NOT RECURSIVELY ENUMERABLE

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ABSTRACT. The set of rationals is not recursive over any field of infinite transcendence degree.

There has been a great deal of work in recursion theory centered around the topic of the existence of recursively enumerable but not recursive sets. Usually the existence of such sets is proved by the construction of a universal set. This method produces sets so far from the realm of ordinary mathematics that at first they might appear to be irrelevant. In this note we shall show that the situation is very different for recursion theory over abstract models.

Consider the recursion theory we get when we start with an arbitrary field of characteristic 0, instead of the integers, as our basic algebraic structure. The recursive functions are those functions we can program in a BASIC-like language using simple variables, assignments, conditionals, and gotos. The assignments use the field operations and constants from the field. The conditionals use $=$, $\neq$, and (if the field is ordered) $<$, $>$. A set is recursively enumerable if it is the domain of a recursive function. This theory was presented in [Fr], [KM], [BSS], and [FM]. (In [KM] Kechris and Moschovakis refer to this concept as prime computability.)

The existence of an r.e. nonrecursive set can be proven very simply in this context. Blum et. al. have pointed out that over the reals every r.e. set has integral Hausdorff dimension. This means, for instance, that the Cantor set and many Julia sets are not r.e. In both these cases it is easily seen that the complement of the Cantor set or any Julia set is r.e. Thus we are able to construct beautiful examples of r.e. but not recursive sets.

Theorem 1. *If K is an Archimedean ordered field or any unordered field of characteristic 0, then the complement of the rationals in K is r.e. iff K has a finite transcendence degree.*

Proof. A condition is called *basic* if it can be defined with a finite conjunction of equalities and inequalities. An effective union of basic conditions is the union of a set of basic conditions whose Gödel numbers form an r.e. set. in

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the usual sense over the integers. We shall use the following fundamental fact about r.e. sets, proven in any of the above references: A set is r.e. over K iff there is a condition $\varphi(x, x_1, \ldots, x_n)$ and a sequence of parameters from K $(k_1, \ldots, k_n)$, such that φ is an effective union of basics and $\varphi(x, k_1, \ldots, k_n)$ defines our set.

Supposing that K has finite transcendence degree, we must show that the irrationals in K are an effective union of basics. By high-school algebra, we may test to see if a given integer polynomial has rational roots. If K were an algebraic field, this would make the irrationals r.e. For x would be irrational iff it is the root of a polynomial with integer coefficients but no rational roots, thus meeting the above criterion for recursive enumerability.

To extend this to the general case, we must consider polynomials whose coefficients are integer polynomials in algebraically independent parameters $k_1, \ldots, k_n$. Such a polynomial $p(x)$ can be regarded as a polynomial in the variables $k_1, \ldots, k_n$ with coefficients in $Q[x]$. By algebraic independence, it vanishes iff all its coefficients vanish. Thus by answering the question of whether a finite set of integer polynomials has a simultaneous rational solution, we can determine whether or not a polynomial in $k_1, \ldots, k_n$ has a rational root. Thus the set of integer polynomials $p(x, x_1, \ldots, x_n)$ such that $p(x, k_1, \ldots, k_n)$ has no rational roots is recursive, and we can proceed as above.

Now suppose that K has infinite transcendence degree. We shall consider only the case where K is an Archimedean ordered field; the unordered case is parallel, but even easier. Suppose that the irrationals in K are r.e. in a finite parameter list. Let k be an element of K not algebraic over these parameters. Then k satisfies some basic condition which puts it into the r.e. set. This basic condition is a finite conjunction of equalities and inequalities, but k satisfies no equalities using the given parameters. Therefore the basic condition defines an open set of reals, and consequently has a rational member: Contradiction.

We should mention that fields of finite transcendence degree do have r.e. but no recursive sets even if the rationals will not work. Archimedean ordered fields of finite transcendence degree have a pairing system, and consequently the usual construction of universal sets can be carried out (see [FM]). For unordered fields, the situation is handled by [FM, Theorem 25]; any r.e. but not recursive set of integers is not recursive over the field.

Using the above method, the reader should easily be able to prove

Theorem 2. *Any closed subset of the reals with an uncountable boundary is not r.e. over the reals.*

Proof. If the boundary is uncountable it must contain a point not algebraic in the parameters. As above, the r.e. set would have to contain an open neighborhood of this boundary point.

Theorem 3. *The Mandelbrot set is co-r.e. but not r.e.*

Proof. The ordinary computer-graphics algorithm to draw the Mandelbrot set M shows that its complement is r.e. To show that M itself is not r.e., we need to know that for every parameter list $x_1, \ldots, x_n$, there is a pair (x, y) on the boundary of M such that the list $x, y, x_1, \ldots, x_n$ satisfies no algebraic equation. But this is well known. Otherwise the boundary of M would be contained in a countable union of curves algebraic in the parameters $x_1, \ldots, x_n$. By the Baire category theorem, there would have to be an open piece of the boundary of M which actually coincided with an algebraic curve, but every open piece of the boundary of M has fractional dimension while algebraic curves have dimension 1.

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