THE INFLUENCE OF A SMALL CARDINAL ON THE PRODUCT OF A LINDELÖF SPACE AND THE IRRATIONALS

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Abstract. It is unknown whether there is in ZFC a Lindelöf space whose product with the irrationals is nonnormal. We give some necessary conditions based on the minimum cardinality of \( a \leq^* \) unbounded family in \( \omega_1 \).

0. Introduction

History. In 1963 [9], E. A. Michael gave the first ZFC example of a normal space and a metric space with a nonnormal product, where the term “space” is an abbreviation for “Hausdorff regular topological space.” (In 1955 [12], M. E. Rudin used a Souslin line to construct a Dowker space—a normal space whose product with the closed unit interval is nonnormal; the existence of a Souslin line was shown to be independent of ZFC in the late 1960s. Rudin constructed a real Dowker space in 1971 [13].) Let \( \mathbb{P}, \mathbb{Q}, \) and \( \mathbb{R} \) denote respectively the irrationals, rationals, and reals with their usual topologies; and let \( \mathbb{M} \) denote the Michael line which is the refinement of \( \mathbb{R} \) obtained by isolating each irrational point. In Michael’s example, \( \mathbb{M} \) is the normal space and \( \mathbb{P} \) is the metric space; \( \mathbb{M} \) is moreover hereditarily paracompact.

A topological space \( X \) is concentrated on a subset \( A \subseteq X \) if for every open set \( U \supseteq A \), \( X \setminus U \) is countable. The Continuum Hypothesis implies the existence of an uncountable subset \( X \subseteq \mathbb{R} \) with \( X \) concentrated on \( \mathbb{Q} \) (A.S. Besicovitch [2], 1934; in fact, \( b = \omega_1 \) is all his proof requires—see Propositions 2 and 3 below). Michael obtained this result independently and pointed out that as a subspace of \( \mathbb{M} \), \( X \) is Lindelöf whereas \( X \times \mathbb{P} \) is nonnormal ([10], 1971; first mentioned in a footnote of [9]).

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Also in the 1963 paper, Michael gave a ZFC example of a Lindelöf space $M'$ and a separable metric space $S$ with $M' \times S$ nonnormal. Let $A$ be a Bernstein set in $\mathbb{R}$; that is, choose $A$ so that each of $A$ and $\mathbb{R} \setminus A$ intersects every uncountable closed set. (A set of this type was first constructed by F. Bernstein in 1908 using transfinite recursion and the Cantor-Bendixson theorem: each of the $\mathfrak{c}$-many closed sets in $\mathbb{R}$ is either countable or has cardinality $\mathfrak{c}$, where $\mathfrak{c} = 2^\omega$; see [11, pp. 23–24].) Note that $\mathbb{R}$ is concentrated on $A$. Let $M'$ be the refinement of $\mathbb{R}$ obtained by isolating each point of $\mathbb{R} \setminus A$, and let $S$ be $\mathbb{R} \setminus A$ with the subspace topology (note that $S$ is not complete).

Is there a real Lindelöf space whose product with $\mathbb{P}$ is nonnormal? This is equivalent to the existence in ZFC of a Lindelöf space and a separable completely metrizable space with a nonnormal product (see Proposition 1 below).

**Michael spaces.** Define a Michael space to be a Lindelöf space whose product with $\mathbb{P}$ is nonnormal. Is there a Michael space in ZFC? In particular, can we build a real Michael space by the method of isolating points in a separable metric space? Define a concentrated Michael space to be a Michael space that is concentrated on a closed subset $A$ where $A \times \mathbb{P}$ is normal. This concept captures the construction of isolating points mentioned above, in which case $A$ is the derived set (i.e., the closed subset of nonisolated points) of the refinement; since $A$ is a (separable) metrizable subspace of the refinement, its product with $\mathbb{P}$ is normal.

In his handbook article ("The Integers and Topology" in [7, pp. 111–167, and in particular, pp. 150–153]), E. K. van Douwen noted some necessary conditions for the existence of a real Michael space which are the point of departure for the current paper. We first establish that regardless of special set-theoretic axioms, a Michael space cannot be constructed by isolating points in an initial space that is completely metrizable (Theorem 1). We then show that the existence of a concentrated Michael space is independent of ZFC (Theorem 2). Finally, we give a lower bound for the cardinality and weight of an arbitrary Michael space (Theorem 3). (The weight of a space $X$, denoted by $w(X)$, is the minimum cardinality of a base for the topology of $X$.) As in the case of the results in the van Douwen article, the cardinal number $\mathfrak{b}$ (defined below) is at the heart of the matter in Theorems 2 and 3.

1. **Propositions**

**Remark.** The propositions below are now more or less standard. We sketch the proofs for the convenience of the reader.

**Proposition 1.** Suppose $X$ and $Y$ are spaces where $Y$ is separable, completely metrizable, and not $\sigma$-compact. Then $X \times Y$ is normal iff $X \times \mathbb{P}$ is normal.

**Proof.** Sufficiency follows from three standard theorems: Normality is preserved under a closed continuous map (see [4, p. 69]); every separable completely metrizable space is the image under a perfect map of a closed subspace of $\mathbb{P}$ (a map is perfect if it is closed and continuous, and each point-inverse set...
is compact); and, a Cartesian product of perfect mappings is perfect [4, p. 237].
(Since \( P \) is homeomorphic to \( \omega P \) [4, p. 348], and every separable completely metrizable space can be embedded as a closed subspace in \( \omega R \) [4, p. 342], we can prove the second theorem by using \( \omega \)-many copies of the Cantor ternary function [7, pp. 203–204] to define a perfect map from a closed subspace of \( P \) onto \( R \), and in turn, from a closed subspace of \( \omega P \) onto \( \omega R \).

Necessity. \( Y \) has a closed subspace homeomorphic to \( P \) (W. Hurewicz; see van Douwen’s proof in [7, pp. 141–142]).

**Notation.** Recall that \( P \) can be identified with \( \omega \omega = \{ f|f: \omega \to \omega \} \) with the product topology: for each \( \xi \in \omega \omega = \{ \eta|\eta: [0, n] \to \omega \text{ for some } n \in \omega \} \), \( \{ f \in \omega \omega: \xi \subseteq f \} \) is a basic open set (see [7, p. 204]). For \( f, g \in \omega \omega \), define \( f \leq g \) if for each \( n \in \omega \), \( f(n) \leq g(n) \); and define \( f \leq^* g \) if there exists \( m \in \omega \) such that for each \( n \geq m \), \( f(n) \leq g(n) \). For each \( g \in \omega \omega \), let \( C(g) = \{ f \in \omega \omega: f \leq g \} \), and let \( C^*(g) = \{ f \in \omega \omega: f \leq^* g \} \).

**Compact sets.** (1) For each \( g \in \omega \omega \), \( C(g) \) is compact and \( C^*(g) \) is \( \sigma \)-compact.

(2) For each \( T \subseteq \omega \omega \), if \( T \) is compact (resp., \( \sigma \)-compact), then there exists \( g \in \omega \omega \) such that \( T \subseteq C(g) \) (resp., \( T \subseteq C^*(g) \)) (Note that the compactness relationship implies the analogue for \( \sigma \)-compactness, since for any sequence \( g: \omega \to \omega \), \( \bigcup_{k \in \omega} C(g_k) \subseteq C^*(h) \) where \( h(n) = \sum_{k \leq n} g_k(n) \).

(3) For all \( f, g \in \omega \omega \), \( C(f) \subseteq C(g) \) (resp., \( C^*(f) \subseteq C^*(g) \)) iff \( f \leq g \) (resp. \( f \leq^* g \)).

**Cardinals.** With respect to \( \leq^* \), let \( b \) be the minimum cardinality of an unbounded family and let \( d \) be the minimum cardinality of a dominant (cofinal) family. Then in ZFC we have:

(1) \( \omega_1 \leq b \leq d \leq \varsigma \);

(2) \( b \) is regular because there is a \( \leq^* \) well-ordered unbounded family of order type \( b \). (To prove this, let \( f: b \to \omega \) with Range(\( f \)) unbounded, and then define \( g: b \to \omega \) recursively so that for all \( \alpha, \beta \in b \) with \( \alpha < \beta \), \( f(\alpha) \leq^* g(\alpha) \leq^* g(\beta) \);

(3) \( b = d \) iff there is a \( \leq^* \) well-ordered dominant family (often called a scale). (To prove sufficiency, note that if \( g: \gamma \to \omega \) is an order-isomorphism onto a dominant subcollection, then \( d \leq \text{cofinality of } \gamma \leq b \); and to prove necessity, let \( f: b \to \omega \) with Range(\( f \)) dominant, and then define \( g \) as in the proof that \( b \) is regular.)

The Continuum Hypothesis implies \( \omega_1 = b = d = \varsigma \), while under Martin’s axiom, \( b = d = \varsigma \) (see [6, p. 87, Example 8]). Since Martin’s axiom is consistent with the negation of the Continuum Hypothesis [6, pp. 278–281], we have that the combinatorial statement \( b = \omega_1 \) is independent of ZFC.

**Consistency results.** In [3], D. K. Burke and S. Davis constructed a Michael space using an \( \omega_1 \)-scale (equivalently, \( d = \omega_1 \)); while in [7], van Douwen
obtained a Michael space under $b = \omega_1$. The hypotheses $c = \omega_1$, $d = \omega_1$, and $b = \omega_1$ are successively weaker statements (see [5]).

**Proposition 2.** Suppose $X$ is an uncountable subset of $\mathbb{P}$. Then the following are equivalent:

1. every uncountable subset of $X$ is $\leq^*$ unbounded in $\omega\omega$;
2. $X$ has countable intersection with every compact subset of $\mathbb{P}$;
3. $X \cup Q$ is concentrated on $Q$;
4. $X \cup Q$ with the subspace topology of $\mathbb{M}$ is Lindelöf.

**Proposition 3.** There is an uncountable subset of $\mathbb{P}$ satisfying each condition of Proposition 2 iff $b = \omega_1$.

**Proof of Proposition 2.** Each of (1) and (2) is equivalent to $X$ having countable intersection with $C^*(g)$ for every $g \in \omega\omega$.

For (2) implies (3), note that a closed set in $\mathbb{R}$ which is contained in $\mathbb{R}\setminus Q$ is a $\sigma$-compact subset of $\mathbb{P}$ and therefore is contained in $C^*(g)$ for some $g \in \omega\omega$. The converse follows from the fact that the complement in $\mathbb{R}$ of a compact subset of $\mathbb{P}$ is an open set in $\mathbb{R}$ containing $Q$.

The equivalence of (3) and (4) is immediate from the definitions.

**Proof of Proposition 3.** Suppose $X$ is an uncountable subset of $\mathbb{P}$ satisfying the first condition of Proposition 2. Let $Y \subseteq X$ with $|Y| = \omega_1$. By hypothesis, $Y$ is unbounded so $b = \omega_1$. For the converse, let $X$ be a $\leq^*$ well-ordered unbounded family in $\omega\omega$ of type $\omega_1$.

2. Theorems

**Lemma 1.** (Rudin and M. Starbird, [14], 1975). The product of a Lindelöf space and a separable metric space is normal iff it is Lindelöf.

**Lemma 2** (Characterization). Suppose $S$ is a separable metric space and $X$ is a Lindelöf space that is concentrated on a closed subset $A \subseteq X$ where $A \times S$ is normal. Then $X \times S$ is normal iff for every uncountable subset $B \subseteq X\setminus A$ and 1–1 function $F: B \to S$, $\text{cl}(\text{Graph}(F)) \cap (A \times S) \neq \emptyset$.

**Proof.** Sufficiency of the contrapositive. We will show that $\text{cl}(\text{Graph}(F))$ and $A \times S$ cannot be separated by disjoint open sets. Let $U$ be an open set in the product with $A \times S \subseteq U$. Let $D$ be a dense countable subset of $S$. For each $d \in D$, let $Y_d = \{x \in X: (x, d) \notin U\}$. By the concentration of $X$ on $A$, each $Y_d$ is countable, and since $B$ is uncountable, we can choose $x_0 \in B \setminus \bigcup_{d \in D} Y_d$. Then $\{x_0\} \times D \subseteq U$, so $(x_0, F(x_0)) \in \text{cl}(U)$.

Necessity of the contrapositive. By Lemma 1 in application to $A$ and $S$, we have that $A \times S$ is Lindelöf. So if every open cover of $A \times S$ also covers each point of all but countably many vertical sections (i.e., the point-inverse sets under projection to $X$), then $X \times S$ is Lindelöf.

Suppose $X \times S$ is nonnormal. Then by the preceding paragraph, we can choose an open set $U \supseteq A \times S$ such that the complement of $U$ intersects uncountably many vertical sections. For each $s \in S$, again let $Y_s = \{x \in X: (x, s) \notin U\}$, and let $Z = \bigcup_{s \in S} Y_s$. Then each $Y_s$ is countable while
Z is uncountable, so we can define \( \sigma: \omega_1 \to Z \) and \( \tau: \omega_1 \to S \) by transfinite recursion so that for each \( \alpha \in \omega_1 \), \( \sigma(\alpha) \in Y_{\tau(\alpha)} \setminus \bigcup_{\beta < \alpha} Y_{\tau(\beta)} \). Let \( B = \text{Range}(\sigma) \) and let \( F = \{(\sigma(\alpha), \tau(\alpha)) : \alpha \in \omega_1 \} \).

**Notation.** Suppose \( S \) is a separable metric space and \( A \subseteq S \) where \( S \) is concentrated on \( A \). Define \( \mathbb{L}(S, A) \) to be the Lindelöf refinement of \( S \) obtained by isolating each point of \( S \setminus A \).

**Corollary 1.** Suppose \( S \) is a separable metric space and \( A \subseteq S \) where \( S \) is concentrated on \( A \) and \( S \setminus A \) is uncountable. Then \( \mathbb{L}(S, A) \times (S \setminus A) \) is nonnormal (where \( S \setminus A \) has the subspace topology).

**Proof.** Use Lemma 2 with \( F \) the identity function on \( S \setminus A \). (This choice of closed sets that cannot be separated is the prototypical construction discovered by Michael.)

**Remark.** In Lemma 3, and consequently in Theorem 1 and Corollary 2 also, \( \mathbb{P} \) can be replaced by any separable completely metrizable space.

**Lemma 3.** Suppose \( Y \) is a separable completely metrizable space, and \( A \subseteq X \subseteq Y \), where \( X \) is concentrated on \( A \) and \( \mathbb{L}(X, A) \times \mathbb{P} \) is nonnormal. Then there is a Cantor set \( K \subseteq Y \) such that \( K \) is disjoint from \( A \) while contained in the \( Y \) closure of \( A \).

**Proof.** Choose \( F \) according to Lemma 2. Let \( G \) be the closure of \( \text{Graph}(F) \) in the topology of \( Y \times \mathbb{P} \). Note that if \( U \) is an open set in \( \mathbb{L}(X, A) \times \mathbb{P} \) containing \( A \times \mathbb{P} \), then there is a set \( V \supseteq A \times \mathbb{P} \) which is open in \( Y \times \mathbb{P} \) and satisfies \( U \supseteq V \cap (X \times \mathbb{P}) \); so if \( U \cap \text{Graph}(F) = \emptyset \), then \( V \cap \text{Graph}(F) = \emptyset \). Thus \( G \) is disjoint from \( A \times \mathbb{P} \).

For the remainder of the proof we will work entirely in the topology of \( Y \times \mathbb{P} \). We need the following theorem: Every uncountable subset \( T \) of a separable metric space contains a condensation point (i.e., there exists \( t \in T \) such that for each open set \( U \) containing \( t \), \( |U \cap T| \geq \omega_1 \)). (For a proof, assume otherwise and use the Lindelöf property on the subspace to obtain a countable open cover of \( T \) where each open set contains only countably many points of \( T \).)

By the theorem just quoted, we can choose a sequence \( \langle U_n : n \in \omega \rangle \) of open sets in the product \( Y \times \mathbb{P} \) such that \( U_0 \) is a basic ("rectangular") open set, and for each \( n > 0 \):

1. \( \text{cl}(U_n) \subseteq U_{n-1} \);
2. each component of \( U_{n-1} \) is a basic open set with diameter \( < \frac{1}{n} \), and contains at least two, but no more than a finite number of, components of \( U_n \);
3. the projections to \( Y \) of the components of \( U_n \) are pairwise disjoint; and
4. each component of \( U_n \) has uncountable intersection with \( \text{Graph}(F) \).
Let $H = \bigcap_{n \in \omega} U_n$. Then by the completeness of $Y \times \mathbb{P}$, $H$ is a Cantor set which is contained in $G$ and intersects uncountably many vertical sections. Let $K$ be the projection of $H$ to $Y$. Then $K \cap A = \emptyset$, while $K$ is contained in the $Y$ closure of $A$ since $X$ is concentrated on $A$ and each neighborhood of each point of $K$ has uncountable intersection with $X$.

**Theorem 1.** Suppose $Y$ is a separable completely metrizable space concentrated on a subset $A$. Then $L(Y, A) \times \mathbb{P}$ is normal (and therefore Lindelöf by Lemma 1) in ZFC.

**Proof.** The concentration of $Y$ on $A$ implies that $A$ intersects every Cantor set contained in $Y$; so by the contrapositive of Lemma 3, $L(Y, A) \times \mathbb{P}$ is normal.

**Corollary 2.** Suppose $A$ is a Bernstein set in $\mathbb{R}$. Then $L(\mathbb{R}, A) \times \mathbb{P}$ is Lindelöf in ZFC.

**Remark.** Suppose $A \subseteq \mathbb{R}$ is Bernstein, and give both $A$ and $\mathbb{R}\setminus A$ the subspace topology. By Corollaries 1 and 2, $L(\mathbb{R}, A) \times \mathbb{P}$ is Lindelöf whereas $L(\mathbb{R}, A) \times \mathbb{R}\setminus A$ is nonnormal. However, completeness in $\mathbb{P}$, or its failure in $\mathbb{R}\setminus A$, is not the only relevant property. The status of $L(\mathbb{R}, A) \times A$ depends upon the choice of $A$. For each $n$ with $n < \omega$ or $n = \omega$, there is a Bernstein set $A$ such that for each $m < n$, $L(\mathbb{R}, A) \times ^mA$ is Lindelöf, whereas $L(\mathbb{R}, A) \times ^nA$ is nonnormal (see [8] for the proof).

**Theorem 2.** There is a concentrated Michael space iff $\mathfrak{b} = \omega_1$.

**Proof.**

Sufficiency. Suppose $\mathfrak{b} = \omega_1$. Use Propositions 2 and 3 to choose an uncountable Lindelöf subspace $X$ of $\mathbb{M}$ with $Q \subseteq X$. Then $X \times \mathbb{P}$ is nonnormal by Lemma 2 using the Michael construction: $S = \mathbb{P}$, $A = \mathbb{Q}$, $B = X\setminus A$, and $F$ is the identity function on $B$.

Necessity. Suppose $X$ is a Lindelöf space that is concentrated on a closed subset $A \subseteq X$ where $A \times \mathbb{P}$ is normal. Suppose further that $X \times \mathbb{P}$ is nonnormal. Choose $F$ according to Lemma 2. Claim. Range($F$) has countable intersection with every compact subset of $\mathbb{P}$. If the claim holds, then $\mathfrak{b} = \omega_1$ by the propositions.

Assume the claim is false and let $T$ be a compact subset of $\mathbb{P}$ and let $G$ be a restriction of $F$ such that $|\text{Graph}(G)| = \omega_1$ and Range($G$) $\subseteq T$. For each $a \in A$, let $W(a)$ be a finite basic open cover of $\{a\} \times T$, where each element is disjoint from cl(Graph($G$)). Let $V = \{\bigcap_{U \in W(a)} \proj_X(U) : a \in A\}$ (where Proj$^X$ is the projection function to $X$). Then $V$ is an open cover of $A$ with Domain($G$) $\cap \bigcup V = \emptyset$, which contradicts the concentration of $X$ on $A$.

**Corollary 3.** A real Michael space cannot be concentrated on a subset whose product with $\mathbb{P}$ is Lindelöf whether or not the subset is closed.

**Proof.** Suppose $X$ is a Lindelöf space concentrated on a subset $A \subseteq X$ where $A \times \mathbb{P}$ is Lindelöf. Let $Y$ be the refinement of $X$ obtained by isolating each
point of \( X \setminus A \). By Theorem 2 and Lemma 1, if \( b > \omega_1 \), then \( Y \times \mathbb{P} \) is Lindelöf; and since \( X \times \mathbb{P} \) has a coarser topology, it too must be Lindelöf.

**Corollary 4.** The derived set of a real Michael space is itself a Michael space.

*Proof.* A Lindelöf space is concentrated on its derived set, so Corollary 3 applies.

**Lemma 4.** Suppose \( X \) is a Lindelöf space. Then for each \( g \in \omega^\omega \), \( X \times C^*(g) \) is a Lindelöf subspace of \( X \times \mathbb{P} \).

*Proof.* The result follows from two facts: (1) the product of a Lindelöf space and a compact space is Lindelöf and (2) \( C^*(g) \) is \( \sigma \)-compact.

**Notation.** Let \( \ell \) be the minimum of \( b \) and \( \text{lub}\{\omega_n : n \in \omega\} \). (We are not using the customary symbol for the above least upper bound to avoid confusion with our use of \( \omega^\omega \) for the function space.)

**Theorem 3.** Suppose \( X \) is a Michael space. Then \( |X| \geq \ell \) and \( w(X) \geq \ell \).

*Proof.* Claim. \( L(X \times \mathbb{P}) \geq \ell \) (where \( L(X \times \mathbb{P}) \) is the Lindelöf degree of \( X \times \mathbb{P} \) which is defined to be the minimum of \( \{\kappa : \text{every open cover of } X \times \mathbb{P} \text{ has a subcover of cardinality } \leq \kappa\} \)). Before proving the claim, first note that the theorem is an immediate consequence since \( L(X \times \mathbb{P}) \leq \min(|X|, w(X)) \), which in turn follows from \( w(X \times \mathbb{P}) = w(X) \) and the fact that each vertical section of \( X \times \mathbb{P} \) is a Lindelöf subspace.

*Proof of the claim.* Let \( \Lambda \) be an open cover of \( X \times \mathbb{P} \) of minimum cardinality which does not have a countable subcover, and let \( \ell = |\Lambda| \). Note that \( L(X \times \mathbb{P}) \geq \ell \). We will show that either \( \ell \) has countable cofinality or \( \ell \geq b \). Thus \( \ell \geq \ell \).

Assume \( \ell \) does not have countable cofinality, and let \( \theta : \ell \to \Lambda \) be a 1–1 correspondence. By transfinite recursion we now define strictly increasing sequences \( \sigma : \ell \to \omega^\omega \) and \( \tau : \ell \to \ell \) (where \( \omega^\omega \) is partially ordered by \( \leq^* \)) such that for each \( \alpha \in \ell \), \( X \times C^*(\sigma_\alpha) \) is not entirely covered by \( \{\theta_\beta : \beta < \text{lub Range}(\tau|\alpha)\} \), but is covered by \( \{\theta_\beta : \beta < \tau_\alpha\} \).

Suppose \( \gamma \in \ell \) and \( \sigma|\gamma \) and \( \tau|\gamma \) have been defined so as to satisfy the above condition. For the extension of \( \sigma \) we consider two cases. Let \( \varepsilon = \text{lub Range}(\tau|\gamma) \). If \( \gamma \) has countable cofinality, then \( \varepsilon < \ell \) by our starting assumption on \( \ell \), so the existence of an appropriate choice for \( \sigma_\gamma \) follows from \( b \geq \omega_1 \) and the minimality in the choice of \( \Lambda \). Assume that \( \gamma \) does not have countable cofinality. Let \( \sigma_\gamma \) be a \( \leq^* \) upper bound for \( \sigma|\gamma \). If \( X \times C^*(\sigma_\gamma) \) is covered by \( \{\theta_\beta : \beta < \varepsilon\} \), then by Lemma 4 there is a countable subcover, which by our assumption on \( \gamma \) implies the existence of \( \delta < \gamma \) such that \( X \times C^*(\sigma_\gamma) \) is covered by \( \{\theta_\beta : \beta < \tau_\delta\} \); this contradicts our hypothesis since \( C^*(\sigma_{\delta+1}) \subseteq C^*(\sigma_\gamma) \). For the extension of \( \tau \), again use Lemma 4 and the uncountable cofinality of \( \ell \).

**Corollary 5.** Suppose \( X \) is a first countable Michael space. Then under Martin’s axiom together with \( \zeta = \omega_n \) for some \( n \in \omega \), \( |X| = w(X) = \zeta \).
**Proof.** By Archangel’skii’s theorem (see [7, p. 19]), \( \mathfrak{c} \) is an upper bound for the cardinality and weight of a first countable Lindelöf space; and since Martin’s axiom implies \( b = \mathfrak{c} \), we have by Theorem 3 that \( \mathfrak{c} \) is also a lower bound.

**Remark.** Recently K. Alster showed in [1] that Martin’s axiom implies the existence of a Michael space. Since \( b = \mathfrak{c} \) under Martin’s axiom and \( \mathfrak{c} \neq \omega_1 \) is consistent with Martin’s axiom, Alster’s example cannot be concentrated. Thus the existence of a Michael space that is not concentrated is consistent with ZFC.

**Problems.** (1) Extend the independence result of Theorem 2 to a larger class of spaces. In particular, is the existence of a Michael space obtained as a refinement of a separable metric space independent of ZFC?

(2) Can Theorem 3 be improved by replacing \( \mathfrak{c} \) with \( b \)?

(3) In both the Michael and Alster examples, one type of point and open set yields the Lindelöf property, while a second type guarantees nonnormality in the product with \( \mathbb{P} \). Is there a homogeneous Michael space in ZFC?

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**References**

13. ____, *A normal space \( X \) for which \( X \times I \) is not normal*, Fund. Math. 73 (1971), 179–186.

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