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RANDOM TIMES AND TIME PROJECTIONS

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ABSTRACT. Random times and their associated time projections are discussed within the context of quantum probability theory. A stochastic integral representation for time projections is obtained, and their order structure is investigated. A quantum analogue of the classical result relating the range and bounded stoppings of the stochastic integral is proved.

INTRODUCTION

In this paper we develop some of the themes introduced in [7], [8] and [5]. Within the context of certain quasi-free representations of the CAR and CCR it was shown, in [7], [8], that one can define the square of an L^2-martingale as a process in an appropriately chosen L^1-space, and this allowed one to prove that such a square has a decomposition into a martingale and an increasing process which is *natural* in the sense defined by Meyer [13]. In addition to this, the *process* obtained by stopping an L^2-martingale was characterized. In [5], it was shown that random times form a complete lattice structure and that this is, to some extent, mirrored by the corresponding family of time projections.

We shall discuss essentially three interelated topics: the structure of the time projections associated with random times, the definition of the stochastic integral with respect to an arbitrary L^2-martingale, and finally, trying the first two parts together, the characterization of the range of the stochastic integral with respect to a given L^2-martingale in terms of the time projections.

Sections 1 and 2 contain a brief review of the notions of process, random time, and the associated time projection. A time projection can be thought of as a generalization of the conditional expectation and also as the operation of stopping a process. This discussion is within the abstract setup.

In §§3, 4, and 5, we take up in detail the quantum stochastic theory of the CAR—although all of the results have obvious analogues within the theory of the CCR. An integral representation theorem for time projections is obtained

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in §3, together with a discussion of their order structure. A construction of the stochastic integral analogous to the classical construction (via the natural increasing part of the Doob–Meyer decomposition) is given in §4. Finally, in §5, the range of the stochastic integral is related to that of the algebra generated by the time projections.

1. Noncommutative processes

The fundamental object within which we work is a noncommutative stochastic base. Specifically, we suppose that we have a von Neumann algebra $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$ and an increasing family of von Neumann subalgebras $\mathfrak{A}_t$ indexed by $[0, \infty]$ such that $\mathfrak{A} \equiv \mathfrak{A}_\infty$ is generated by the collection $\mathfrak{A}_t\colon t \in [0, \infty)$. We suppose, further, that there is a unit vector $\Omega \in \mathscr{H}$ cyclic and separating for $\mathfrak{A}$ and a family $E_t\colon t \in [0, \infty]$ of normal ω-invariant conditional expectations $E_t\colon \mathfrak{A} \to \mathfrak{A}_t$, where $\omega(\cdot) = (\cdot\Omega, \Omega)$.

Since Ω is cyclic and separating, the map $x \mapsto \|x\Omega\|$, $x \in \mathfrak{A}_t$, is a norm on $\mathfrak{A}_t$ and the completion is a Hilbert subspace $\mathscr{H}_t$ of $\mathscr{H}$, with $\mathscr{H}_\infty = \mathscr{H}$. We denote the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_t$ by P_t. It is then easy to see that $P_t x\Omega = E_t(x)\Omega$ for $x \in \mathfrak{A}$. By analogy with the classical (commutative) theory, we sometimes write $L^2(\mathfrak{A}_t)$ for $\mathscr{H}_t$, $L^2(\mathfrak{A})$ for $\mathscr{H}$, $L^\infty(\mathfrak{A}_t)$ for $\mathfrak{A}_t$, and $L^\infty(\mathfrak{A})$ for $\mathfrak{A}$ (in these last two cases, the $\|\cdot\|_\infty$ norm is just the operator norm).

We define $L^1(\mathfrak{A})$ to be the completion of $\mathfrak{A}$ with respect to the norm

$$x \mapsto \sup\{(y\Omega, Jx\Omega)\colon y \in \mathfrak{A}, \|y\| = 1\},$$

where J denotes the conjugate-linear unitary operator as given by the modular theory of $(\mathfrak{A}, \Omega)$. Then $L^1(\mathfrak{A}) \simeq \mathfrak{A}_*$. (For full details of this, see [7]). $L^1(\mathfrak{A}_t)$ is the closure of $\mathfrak{A}_t$ in $L^1(\mathfrak{A})$; thus $L^1(\mathfrak{A}_t) \subseteq L^1(\mathfrak{A})$, $t \in [0, \infty]$. One has $\mathfrak{A}_t \subseteq L^2(\mathfrak{A}_t) \subseteq L^1(\mathfrak{A}_t)$, for $t \in [0, \infty]$, and the conditional expectations E_t extend to bounded maps $L^2(\mathfrak{A}) \to L^2(\mathfrak{A}_t)$ and $L^1(\mathfrak{A}) \to L^1(\mathfrak{A}_t)$, which we continue to denote by E_t. (As remarked above, the extension of E_t to L^2 is just P_t.)

A process is a family X_t indexed by $t \in [0, \infty)$ such that $E_t X_t = X_t$ for all t. A process X_t is a martingale if $E_s X_t = X_s$ for all $0 \le s \le t < \infty$. Thus, we have the notion of $\mathfrak{A}$-valued, $L^2(\mathfrak{A})$-valued or $L^1(\mathfrak{A})$-valued processes and martingales. We will also think of a process as a function on $[0, \infty)$.

If $\zeta \in \mathscr{H}$, then $(P_t\zeta)$ is a bounded $\mathscr{H}$-valued martingale. Every bounded $\mathscr{H}$-valued martingale has this form.

Proposition 1.1. *Let (ζ_t) be a bounded $\mathscr{H}$-valued martingale. Then there is a $\zeta \in \mathscr{H}$ such that $\zeta_t = P_t\zeta$ for all $t \in [0, \infty)$.*

Proof. Let $K = \sup\{\|\zeta_t\| : t \in [0, \infty)\}$. Then for fixed $\xi \in \mathscr{H}$ and $0 \le m \le n < \infty$,

$$\begin{aligned} |(\zeta_n - \zeta_m, \xi)| &= |((P_n - P_m)\zeta_n, \xi)| \\ &= |(\zeta_n, (P_n - P_m)\xi)| \\ &\le K\|(P_n - P_m)\xi\| \\ &\to 0 \quad \text{as } m, n \to \infty \end{aligned}$$

since $P_n \uparrow \mathbf{1}$.

Hence (ζ_n) is weakly Cauchy, and hence there is a $\zeta \in \mathscr{H}$ such that $\zeta_n \to \zeta$ weakly as $n \to \infty$. Thus, for $n > t$, $\zeta_t = P_t\zeta_n$ converges weakly to $P_t\zeta$ as $n \to \infty$, giving $\zeta_t = P_t\zeta$. □

Note that it follows that $\zeta_t \to \zeta$ strongly as $t \to \infty$. Thus every bounded $\mathscr{H}$-valued martingale has a 'natural' value at infinity, $\zeta_\infty \equiv \zeta = \lim \zeta_t$. Such martingales are determined completely by this value—hence we have a one-to-one correspondence between elements of $\mathscr{H}$ and bounded $\mathscr{H}$-valued martingales.

Our concern in this paper is with the noncommutative filtrations and processes for the quantum stochastic theory of the CAR and CCR [3] (for the case when the defining quasi-free state has no Fock part). As noted in [7], this also includes the Itô–Clifford theory of [2].

2. Time projections

We recall the definitions of random time and its associated time projection. The motivation for these definitions is given in [1], [4], [5].

Definition 2.1. A random time τ, adapted to the filtration $\{\mathfrak{A}_t : t \in [0, \infty]\}$, is an increasing projection-valued process (q_t), $t \in [0, \infty]$, with $q_0 = 0$ and $q_\infty = \mathbf{1}$.

Thus q_t is a projection in $\mathfrak{A}_t$ for all $t \in \mathbf{R}^+$ and $q_s \le q_t$ whenever $0 \le s \le t \le \infty$. Note that we do not require that $\sup q_t = \mathbf{1}$ as in [5].

A random time is the noncommutative counterpart to a stopping time in the classical theory.

Definition 2.2. A random time $\tau = (q_t)$ is called *simple* if it assumes only finitely many distinct values; thus $\tau = (q_t)$ is simple if and only if there is some finite subdivision $0 = t_0 < t_1 < \cdots < t_n = \infty$ of $[0, \infty]$ such that

$$q_t = \sum_{i=0}^{n-1} q_{t_i} \chi_{[t_i, t_{i+1})}(t)$$

for $t \in \mathbf{R}^+$.

Proposition 2.3. *For any given random time $\tau = (q_t)$, there is a sequence $\tau_n = (q_t^{(n)})$, $n = 1, 2, \ldots$ of simple random times such that $q_t^{(n)} \uparrow q_t$ as $n \to \infty$ for (Lebesgue) almost every $t \in \mathbf{R}^+$.*

Proof. The function $t \mapsto (q_t\Omega, \Omega)$ is bounded and increasing on $\mathbf{R}^+$ and so has at most countably many jumps. Since Ω is separating, it follows that $t \mapsto q_t$ is (Lebesgue) almost everywhere strongly continuous on $\mathbf{R}^+$. For each $n = 1, 2, \ldots$ put

$$q_t^{(n)} = \sum_{k=0}^{n2^n-1} q_{k/2^n}\chi_{[k/2^n, (k+1)/2^n)}(t) + \mathbf{1}\chi_{[n,\infty]}(t).$$

Then $\tau_n = (q_t^{(n)})$ is a simple random time, for each n, and evidently $q_t^{(n)} \uparrow q_t$ as $n \to \infty$ for (Lebesgue) almost every $\mathbf{t} \in \mathbf{R}^+$. □

Remark. By explicitly taking into account the jumps of (q_t), one can prove that there is a sequence of simple random times which converge to τ everywhere on $\mathbf{R}^+$; see [1].

Definition 2.4. Let $\tau = (q_t)$ be a random time and let $\theta = \{0 = t_0, t_1, \ldots, t_n = \infty\}$, $t_0 < t_1 < \cdots < t_n$, be a finite subdivision of $[0, \infty]$. The simple random time associated with τ and θ is that given by $\tau(\theta) = (q_t^\theta)$, where

$$q_t^\theta = \sum_{i=0}^{n-1} q_{t_i}\chi_{[t_i, t_{i+1})}(t)$$

for $t \in [0, \infty)$, and $q_\infty^\theta = \mathbf{1}$.

If the family Θ of finite subdivisions of $[0, \infty]$ is partially ordered by refinement, then, as in Proposition 2.3, it follows that the net $\{\tau(\theta) = (q_t^\theta) : \theta \in \Theta\}$ converges strongly (Lebesgue) almost everywhere to $\tau = (q_t)$.

Definition 2.5. Let $\tau' = (q_t')$ and $\tau'' = (q_t'')$ be random times. We say that $\tau' \le \tau''$ if and only if $q_t' \ge q_t''$ for all $t \in \mathbf{R}^+$. We define $\tau' \wedge \tau''$ and $\tau' \vee \tau''$ to be the random times $\tau' \wedge \tau'' = (q_t' \vee q_t'')$ and $\tau' \vee \tau'' = (q_t' \wedge q_t'')$.

Thus the family of random times is partially ordered and forms a complete lattice [5].

Definition 2.6. Let (ζ_t) be an L^2-process and $\tau = (q_t)$ a random time. For $\theta = \{0 = t_0, \ldots, t_n = \infty\} \in \Theta$, we define

$$\zeta_{\tau(\theta)} = \sum_{i=0}^{n-1}(q_{t_{i+1}} - q_{t_i})\zeta_{t_{i+1}} \equiv \sum_\theta \Delta q_{t_i}\zeta_{t_{i+1}}.$$

If the net $(\zeta_{\tau(\theta)} : \theta \in \Theta)$ converges in L^2 (i.e., in $\mathscr{H}$) then we denote the limit by ζ_τ and call it the *stopped process* or ζ *stopped by* τ. (Similarly, one can define stopped L^1-processes using the action of L^∞ on L^1 (see [7]), but we shall be concerned here only with stopped L^2-processes.)

If (ζ_t) is an L^2-bounded martingale, then $\zeta_t = P_t\zeta$ for some $\zeta \in \mathscr{H}$, and so $\zeta_{\tau(\theta)}$ becomes

$$\begin{aligned}\zeta_{\tau(\theta)} &= \sum_\theta \Delta q_{t_i} P_{t_{i+1}}\zeta \\ &= M_{\tau(\theta)}\zeta,\end{aligned}$$

where $M_{\tau(\theta)} = \sum_{i=0}^{n-1}(q_{t_{i+1}} - q_{t_i})P_{t_{i+1}}$. For each $\theta \in \Theta$, $M_{\tau(\theta)}$ is a bounded linear operator on $\mathscr{H}$.

Theorem 2.7. *For each random time τ and finite subdivision $\theta \in \Theta$, the operator $M_{\tau(\theta)}$ is an orthogonal projection on $\mathscr{H}$, and if σ is a random time with $\tau \leq \sigma$, then $M_{\tau(\theta)} \leq M_{\sigma(\theta)}$. If θ_2 is a refinement of θ_1 in Θ, then $M_{\tau(\theta_2)} \leq M_{\tau(\theta_1)}$.*

Proof. See [5]. □

It follows from this theorem that for each random time τ the net $\{M_{\tau(\theta)} : \theta \in \Theta\}$ converges (strongly) to an orthogonal projection on $\mathscr{H}$, which we denote by M_τ.

Definition 2.8. For each random time τ, the orthogonal projection M_τ is called the *time projection* associated with τ.

As an immediate corollary to Theorem 2.7, we have the following:

Theorem 2.9 (Optimal Stopping). *For random times τ, σ with $\tau \leq \sigma$, we have $M_\tau \leq M_\sigma$.*

Proof. One just takes limits in Theorem 2.7 [5]. □

If $\zeta \in \mathscr{H}$, then Theorem 2.9 gives $M_\tau M_\sigma \zeta = M_\tau \zeta$. In "process" terms this becomes $(\zeta_\sigma)_\tau = \zeta_\tau$; i.e., optional stopping. (For more details see [4, 5].)

We shall obtain a converse to this theorem in §3 (in the proof of Corollary 3.5).

Each deterministic time $s \in [0, \infty)$ defines a random time $\hat{s} = (q_t)$, say, by setting $q_t = 0$ for $0 \leq t \leq s$, $q_t = \mathbf{1}$ for $s < t$. Then evidently $M_s = P_s$, and so we can regard time projections M_τ as generalizing the conditional expectation.

Let $\tau = (q_t)$ be a random time, $\theta \in \Theta$ and $\zeta \in \mathscr{H}$. Then we have

$$\begin{aligned} M_{\tau(\theta)}\zeta &\equiv \sum_\theta \Delta q_{t_i} P_{t_{i+1}} \zeta = \sum_{i=0}^{n-1} (q_{t_{i+1}} - q_{t_i}) P_{t_{i+1}} \zeta \\ &= q_{t_n} P_{t_n} \theta - q_{t_0} P_{t_0} \zeta - \sum_{i=0}^{n-1} q_{t_i} (P_{t_{i+1}} - P_{t_i}) \zeta \\ &= \zeta - \sum_{i=0}^{n-1} q_{t_i} (P_{t_{i+1}} - P_{t_i}) \zeta \\ &\equiv \zeta - \sum_\theta q_{t_i} \Delta P_{t_i} \zeta, \end{aligned}$$

where we have used $q_0 = 0$, $q_\infty = \mathbf{1}$ (in $\mathfrak{A}$), and $P_\infty = \mathbf{1}$ in $\mathscr{B}(\mathscr{H})$. The sum $\sum_\theta q_{t_i} \Delta P_{t_i} \zeta$ is the stochastic integral of the simple $\mathfrak{A}$-valued integrand q_t^θ with respect to the L^2-valued martingale $\zeta_t = P_t \zeta$. Accordingly, we shall write $M_\tau \zeta$ as

$$M_\tau \zeta = \zeta - \int_0^\infty q_t \, d\zeta_t,$$

where the integral exists as a limit, in $\mathscr{H}$, of the above Riemann sums, since $M_{\tau(\theta)} \to M_\tau$.

But

$$\sum_\theta \Delta P_{t_i}\zeta = \zeta - P_0\zeta,$$

and so we can write $M_{\tau(\theta)}\zeta$ as

$$M_{\tau(\theta)}\zeta = P_0\zeta + \sum_\theta q_{t_i}^\perp \Delta P_{t_i}\zeta$$

and thus we may write

$$M_\tau\zeta = P_0\zeta + \int_0^\infty q_t^\perp \, d\zeta_t,$$

where, as above, the integral is the strong limit of Riemann sums.

We shall consider stochastic integral representations further in the next section (Theorem 3.2 and Corollary 3.3).

Our next result shows that the time projections associated with simple random times form a lattice.

Theorem 2.10. *Let $\tau = (p_t)$ and $\sigma = (q_t)$ be random times, and let $\theta \in \Theta$. Then*

$$M_{(\sigma\vee\tau)(\theta)} = M_{\sigma(\theta)} \vee M_{\tau(\theta)}$$

and

$$M_{(\sigma\wedge\tau)(\theta)} = M_{\sigma(\theta)} \wedge M_{\tau(\theta)}.$$

Proof. Suppose that $\theta = \{0 = t_0, \dots, t_n = \infty\} \in \Theta$. Then for any $\zeta \in \mathscr{H}$, we have

$$M_{\sigma(\theta)}^\perp\zeta = \sum_\theta q_{t_i}\Delta P_{t_i}\zeta = \sum_{i=0}^{n-1} q_{t_i}(P_{t_{i+1}} - P_{t_i})\zeta$$

and hence

$$\begin{aligned} M_{\tau(\theta)}^\perp M_{\sigma(\theta)}^\perp\zeta &= \sum_{j=1}^{n-1} p_{t_j}\Delta P_{t_j}\left(\sum_{i=0}^{n-1} q_{t_i}\Delta P_{t_i}\zeta\right) \\ &= \sum_{j=0}^{n-1}\sum_{i=0}^{n-1} p_{t_j}\Delta P_{t_j} q_{t_i}\Delta P_{t_i}\zeta \\ &= \sum_{j=0}^{n-1}\sum_{i=0}^{n-1} p_{t_j}\Delta E_{t_j}(q_{t_i})\Delta P_{t_j}\Delta P_{t_i}\zeta, \end{aligned}$$

where $\Delta E_{t_j} = E_{t_{j+1}} - E_{t_j}$ and the conditional expectation is

$$E_t = \sum_{j=0}^{n-1} p_{t_j} q_{t_j}\Delta P_{t_j}\zeta$$

since $\Delta P_{t_i}\Delta P_{t_j} = 0$ if $i \neq j$.

It follows that

$$(M_{\tau(\theta)}^{\perp} M_{\sigma(\theta)}^{\perp})^k \zeta = \sum_{j=0}^{n-1} (p_{t_j} q_{t_j})^k \Delta P_{t_j} \zeta$$

for any $k = 1, 2, \ldots$.

Letting $k \to \infty$, we obtain

$$\begin{aligned} M_{\tau(\theta)}^{\perp} \wedge M_{\sigma(\theta)}^{\perp} \zeta &= \sum_{j=0}^{n-1} p_{t_j} \wedge q_{t_j} \Delta P_{t_j} \zeta \\ &= M_{(\tau \vee \sigma)(\theta)} \zeta . \end{aligned}$$

Taking orthogonal complements, we see that

$$M_{\tau(\theta)} \vee M_{\sigma(\theta)} = M_{\tau(\theta) \vee \sigma(\theta)} .$$

For the infimum, we begin with

$$M_{\sigma(\theta)} \zeta = P_0 \zeta + \sum_{i=0}^{n-1} q_{t_i}^{\perp} \Delta P_{t_i} \zeta .$$

As above, we see that

$$(M_{\tau(\theta)} M_{\sigma(\theta)})^k \zeta = P_0 \zeta + \sum_{i=0}^{n-1} (p_{t_i}^{\perp} q_{t_i}^{\perp})^k \Delta P_{t_i} \zeta .$$

Letting $k \to \infty$, we get

$$\begin{aligned} M_{\tau(\theta)} \wedge M_{\sigma(\theta)} \zeta &= P_0 \zeta + \sum_{i=0}^{n-1} p_{t_i}^{\perp} \wedge q_{t_i}^{\perp} \Delta P_{t_i} \zeta \\ &= M_{(\tau \wedge \sigma)(\theta)} \zeta , \end{aligned}$$

as required. □

Corollary 2.11. *For any random times τ, σ we have*

$$M_\tau \wedge M_\sigma = M_{\tau \wedge \sigma} .$$

Proof. We have $\tau \wedge \sigma \le \tau$ and so $M_{\tau \wedge \sigma} \le M_\tau$ (by Optional Stopping, Theorem 2.9). Similarly, $M_{\tau \wedge \sigma} \le M_\sigma$. Hence $M_{\tau \wedge \sigma} \le M_\tau \wedge M_\sigma$. On the other hand, for any $\theta \in \Theta$,

$$M_{\tau(\theta)} \wedge M_{\sigma(\theta)} = M_{(\tau \wedge \sigma)(\theta)}$$

by Theorem 2.10. Hence $M_\tau \wedge M_\sigma \le M_{\tau(\theta)} \wedge M_{\sigma(\theta)} = M_{(\tau \wedge \sigma)(\theta)}$, for all $\theta \in \Theta$, giving $M_\tau \wedge M_\sigma \le M_{\tau \wedge \sigma}$, from which the result follows. □

It is not clear whether the corresponding result for the suprema holds in general. We will see, in the next section, that it is true for the quasi-free setup, where one has integral formulae for the various time projections.

The final result of this section concerns the range of the time projection.

Theorem 2.2. *Let $\tau = (p_t)$ be a stopping time. Then $\zeta \in L^2(\mathfrak{A})$ is in the range of the time projection M_τ if and only if $p_t\zeta \in L^2(\mathfrak{A}_t)$ for all $t \in [0, \infty]$.*

Proof. If $\zeta = M_\tau \zeta$, then

$$\begin{aligned} p_t\zeta = p_t M_\tau \zeta &= \lim_\theta p_t \sum \Delta p_s \zeta_s \\ &= \lim_\theta \sum p_t \Delta p_s \zeta_s, \end{aligned}$$

which belongs to $L^2(\mathfrak{A}_t)$ since $\zeta_s \in L^2(\mathfrak{A}_t)$ for all $s \le t$, and $p_t \Delta p_s = p_t \wedge p_{s_{i+1}} - p_t \wedge p_{s_i} \in L^\infty(\mathfrak{A}_t)$.

Hence $p_t\zeta \in L^2(\mathfrak{A}_t)$ for every $t \in [0, \infty]$.

Conversely, if $p_t\zeta \in L^2(\mathfrak{A}_t)$ for all $t \in [0, \infty]$, then

$$\begin{aligned} M_{\tau(\theta)}\zeta &= \sum \Delta p_s \zeta_s \\ &= \sum \Delta p_s p_{s_{i+1}} \zeta_{s_{i+1}} \\ &= \sum \Delta p_s P_{s_{i+1}} (p_{s_{i+1}} \zeta) \\ &= \sum \Delta p_s p_{s_{i+1}} \zeta \\ &= \sum \Delta p_s \zeta \\ &= \zeta. \end{aligned}$$

Taking the limit gives $M_\tau \zeta = \zeta$. □

3. The quantum stochastic theory of the CAR

We shall specialize now take up in detail the CAR theory as developed in [3]. Consider the gauge-invariant quasi-free representation of the CAR over $L^2(\mathbf{R}^+)$ given by the state ω with

$$\omega(b^*(f)b(g)) = \int_0^\infty f(s)\overline{g(s)}\rho(s)\,ds,$$

where $0 < \rho < 1$ almost everywhere, and $f, g \in L^2(\mathbf{R}^+)$.

Let $\mathscr{A}$ denote the CAR von Neumann algebra realized concretely on $\mathscr{H}$, the tensor product of two copies of the antisymmetric Fock space over $L^2(\mathbf{R}^+)$, so that ω is the vector state $\omega(\cdot) = (\cdot\Omega, \Omega)$, where $\Omega = \Omega_0 \otimes \Omega_0$, Ω_0 being the Fock vacuum vector. The conditions on ρ imply that Ω is cyclic and separating for $\mathscr{A}$ on $\mathscr{H}$. For $t \ge 0$, let $\mathscr{A}_t$ denote the von Neumann subalgebra of $\mathscr{A}$ generated by the operators $\{b^*(f)\colon \operatorname{supp} f \subset [0, t]\}$. Then there exist ω-invariant normal conditional expectations $e_t\colon \mathscr{A} \to \mathscr{A}_t$, and these extend to conditional expectations $E_t\colon L^1(\mathscr{A}) \to L^1(\mathscr{A}_t)$ as described in §1. Let $b_t = b(\chi_{[0,t]})$. Then b_t and b_t^* are $L^\infty(\mathscr{A})$-valued martingales, and one can construct quantum stochastic integrals $\int_0^t db_s^* \xi(s)$ and $\int_0^t db_s \eta(s)$ for suitable adapted $\mathscr{H}$-valued integrands ξ and η. These stochastic integrals are themselves centered $\mathscr{H}$-valued martingales and obey isometry relations. (For the details of these results see [3, 7]).

There is a converse—the martingale representation theorem: every $\mathscr{H}$-valued martingale ζ_t, say, can be written as

$$\zeta_t = \alpha\Omega + \int_0^t db_s^* \xi(s) + \int_0^t db_s \eta(s)$$

for suitable $\alpha \in \mathbf{C}$ and integrands ξ and η; and this representation is unique [11, 14]. (The uniqueness is an easy consequence of the isometry relations.) There is a similar martingale representation theorem for bosons (see [9, 12, 15]).

We will use the stochastic integral representation of elements of $\mathscr{H}$ (see [14, 15]) to give corresponding representations for the action of the time projections M_τ.

Theorem 3.1. *For any* $\zeta \in \mathscr{H}$, *there is unique* $\alpha \in \mathbf{C}$ *and processes* $\xi \in L^2(\mathbf{R}^+, (1-\rho(s))\,ds; \mathscr{H})$ *and* $\eta \in L^2(\mathbf{R}^+, \rho(s)\,ds; \mathscr{H})$ *such that*

$$\zeta = \alpha\Omega + \int_0^\infty db_s^* \xi(s) + \int_0^\infty db_s \eta(s).$$

Proof. See [14]. □

We will write $\zeta = (\alpha, \xi, \eta)$ for notational convenience.

Theorem 3.2. *Let* $\tau = (q_t)$ *be a random time and let* $\zeta = (\alpha, \xi, \eta) \in \mathscr{H}$. *Then*

$$M_\tau \zeta = \alpha\Omega + \int_0^\infty db_s^* \beta(q_s^\perp)\xi(s) + \int_0^\infty db_s \beta(q_s^\perp)\eta(s),$$

where β *is the (spatial) parity automorphism of* $\mathfrak{A}$.

Proof. Let $(\tau_n) = ((q_t^{(n)}))$ be any sequence of simple random times such that $q_t^{(n)}$ converges strongly to q_t (Lebesgue) almost everywhere, and let (ξ_m) and (η_m) be sequences of simple $\mathscr{H}$-valued processes converging to ξ and η in $L^2(\mathbf{R}^+, (1-\rho(s))\,ds; \mathscr{H})$ and $L^2(\mathbf{R}^+, \rho(s)\,ds; \mathscr{H})$, respectively.

Then $\zeta_m = (\alpha, \xi_m, \eta_m)$ converges to ζ in $\mathscr{H}$ as $m \to \infty$ (this is a direct consequence of the isometry relations satisfied by the stochastic integrals [3]).

Now

$$M_{\tau_n} \zeta_m = P_0 \zeta_m + \int_0^\infty q_t^{(n)\perp} d\zeta_m(t),$$

where the integral on the right-hand side is actually a finite sum. By writing out this sum explicitly and using the definition of the stochastic integrals for simple integrands, one readily sees that

$$M_{\tau_n} \zeta_m = \alpha\Omega + \int_0^\infty db_s^* \beta(q_s^{(n)\perp})\xi_m(s) + \int_0^\infty db_s \beta(q_s^{(n)\perp})\eta_m(s).$$

As $n \to \infty$, the strong convergence of τ_n implies that $\beta(q_s^{(n)\perp}\xi_m(s) \to \beta(q_s^{\perp})\xi_m(s)$ in $L^2(\mathbf{R}^+, (1-\rho(s))\,ds; \mathscr{H})$ and that $\beta(q_s^{(n)\perp})\eta_m(s) \to \beta(q_s^{\perp})\eta_m(s)$ in $L^2(\mathbf{R}^+, \rho(s)\,ds; \mathscr{H})$, and so

$$M_{\tau_n}\zeta_m \to (\alpha, \beta(q_\cdot^{\perp})\xi_m, \beta(q_\cdot^{\perp})\eta_m)$$

in $\mathscr{H}$. The limit does not depend on the particular sequence (τ_n). Since the net $\{M_{\tau(\theta)}\zeta_m : \theta \in \Theta\}$ converges to $M_\tau\zeta_m$ in $\mathscr{H}$, it follows that

$$M_\tau\zeta_m = (\alpha, \beta(q_\cdot^{\perp})\xi_m, \beta(q_\cdot^{\perp})\eta_m).$$

Now, letting $m \to \infty$, $\beta(q_\cdot^{\perp})\xi_m(\cdot) \to \beta(q_\cdot^{\perp})\xi(\cdot)$ in $L^2(\mathbf{R}^+, (1-\rho(s))\,ds; \mathscr{H})$ and $\beta(q_\cdot^{\perp})\eta_m(\cdot) \to \beta(q_\cdot^{\perp})\eta(\cdot)$ in $L^2(\mathbf{R}^+, \rho(s)\,ds; \mathscr{H})$, and so

$$M_\tau\zeta = \lim M_\tau\zeta_m = (\alpha, \beta(q_\cdot^{\perp})\xi, \beta(q_\cdot^{\perp})\eta),$$

and the proof is complete. □

Corollary 3.3. *For any random time $\tau = (q_t)$ and any $\zeta = (\alpha, \xi, \eta)$ in $\mathscr{H}$,*

$$M_\tau^{\perp}\zeta = \int_0^\infty db_s^*\,\beta(q_s)\xi(s) + \int_0^\infty db_s\,\beta(q_s)\eta(s).$$

Proof. This follows immediately from the two previous theorems. □

Corollary 3.4. *The map $\tau \mapsto M_\tau$ is continuous in the sense that if (τ_n) is any sequence of random times which converge (Lebesgue) almost everywhere to the random time τ, then (M_{τ_n}) converges strongly to M_τ.*

Proof. This follows immediately from the stochastic integral formula above and dominated convergence. □

We now turn to the order structure.

Theorem 3.5. *Let τ, σ be random times. Then*

$$M_{\sigma\vee\tau} = M_\sigma \vee M_\tau \quad \textit{and} \quad M_{\sigma\wedge\tau} = M_\sigma \wedge M_\tau.$$

Proof. Suppose that $\tau = (q_t)$ and $\sigma = (p_t)$. Let $\zeta = (\alpha, \xi, \eta) \in \mathscr{H}$. Then, by Theorem 3.2,

$$(M_\sigma M_\tau)^n\zeta = (\alpha, \beta((p_\cdot^{\perp}q_\cdot^{\perp})^n)\xi, \beta((p_\cdot^{\perp}q_\cdot^{\perp})^n)\eta).$$

As $n \to \infty$, $(p_s^{\perp}q_s^{\perp})^n \to p_s^{\perp} \wedge q_s^{\perp}$ strongly for each $s \in [0, \infty)$ and so, by dominated convergence,

$$\beta((p_\cdot^{\perp}q_\cdot^{\perp})^n)\xi(\cdot) \to \beta(p_\cdot^{\perp} \wedge q_\cdot^{\perp})\xi(\cdot) = \beta((p_\cdot \vee q_\cdot)^{\perp})\xi(\cdot)$$

in $L^2\mathbf{R}^+, (1-\rho(s))\,ds; \mathscr{H})$, and

$$\beta((p_\cdot^{\perp}q_\cdot^{\perp})^n)\eta(\cdot) \to \beta(p_\cdot^{\perp} \wedge q_\cdot^{\perp})\eta(\cdot) = \beta((p_\cdot \vee q_\cdot)^{\perp})\eta(\cdot)$$

in $L^2(\mathbf{R}^+, \rho(s)\,ds; \mathscr{H})$. It follows that $(M_\sigma M_\tau)^n\zeta \to M_{\sigma\wedge\tau}\zeta$ in $\mathscr{H}$ as $n \to \infty$. But $(M_\sigma M_\tau)^n$ converges strongly to $M_\sigma \wedge M_\tau$, and the result follows.

An entirely similar argument, using Corollary 3.3, shows that

$$\begin{aligned} M_\sigma^\perp \wedge M_\tau^\perp \zeta &= \int_0^\infty db_s^* \beta(p_s \wedge q_s)\xi(s) + \int_0^\infty db_s \beta(p_s \wedge q_s)\eta(s) \\ &= M_{\sigma\vee\tau}^\perp \zeta \end{aligned}$$

for any $\zeta \in \mathscr{H}$. That is, $M_\sigma^\perp \wedge M_\tau^\perp = M_{\sigma\vee\tau}^\perp$, and so $M_\sigma \vee M_\tau = M_{\sigma\vee\tau}$. □

Corollary 3.6. *The lattice of time projections is complete, and the map* $\tau \to M_\tau$ *is an order sequentially continuous lattice morphism: that is, for any family* $\mathscr{F} = \{M_\tau : \tau \in \Lambda\}$ *of time projections,* $\sup \mathscr{F}$ *and* $\inf \mathscr{F}$ *are time projections, and if* Λ *is countable, then* $\sup \mathscr{F} = M_{\sup\Lambda}$ *and* $\inf \mathscr{F} = M_{\inf\Lambda}$.

Proof. We first observe that since $\mathscr{H}$ is separable, the strong operator topology on the closed unit ball of $\mathscr{B}(\mathscr{H})$ is metrizable. Hence there is an increasing sequence (M_{τ_n}) in $\mathscr{F}$ converging strongly to $\sup \mathscr{F}$.

For each n, let $\sigma_n = \bigvee_{m\le n} \tau_m$. Then (σ_n) is an increasing sequence of random times with $M_{\tau_n} \le M_{\sigma_n} = \bigvee_{m\le n} M_{\tau_m}$ for all n, by Theorems 2.9 and 3.5. Hence (M_{σ_n}) increases to $\sup \mathscr{F}$. Let $\zeta = (\alpha, \xi, \eta) \in \mathscr{H}$. Then by Theorem 3.2, $M_{\sigma_n}\zeta = (\alpha, \beta(q_\cdot^{(n)\perp})\xi(\cdot), \beta(q_\cdot^{(n)\perp}\eta(\cdot))$, where $\sigma_n = (q_t^{(n)})$.

Write $\tau_m = (p_t^{(m)})$. Then $q_t^{(n)\perp} = \bigvee_{m\le n} p_t^{(m)\perp}$ increases to $\bigvee_m p_t^{(m)\perp} = p_t^\perp$, say, as $n \to \infty$, for each t. It follows, by dominated convergence, that $M_{\sigma_n}\zeta \to (\alpha, \beta(p_\cdot^\perp)\xi(\cdot), \beta(p_\cdot^\perp)\eta(\cdot))$ as $n \to \infty$. Thus, by Theorem 3.2, $\sup \mathscr{F} = M_\sigma$ where $\sigma = (p_t) = \bigvee_m \tau_m$.

Now $\sup \mathscr{F} = M_\sigma$, and so $M_\tau \le M_\sigma$ for each $\tau \in \Lambda$; i.e., $M_\sigma M_\tau = M_\tau$ for each $\tau \in \Lambda$. Let $\zeta = (\alpha, \xi, \eta) \in \mathscr{H}$. Then $M_\sigma M_\tau \zeta = M_\tau \zeta$ gives

$$(\alpha, \beta(p_\cdot^\perp q_\cdot^\perp)\xi(\cdot), \beta(p_\cdot^\perp q_\cdot^\perp)\eta(\cdot)) = (\alpha, \beta(q_\cdot^\perp)\xi(\cdot), \beta(q_\cdot^\perp)\eta(\cdot)),$$

where $\sigma = (p_t)$, $\tau = (q_t)$.

Taking $\xi(t) = e^{-t}\Omega \in L^2(\mathbf{R}^+, (1-\rho(s))\,ds; \mathscr{H})$, we deduce that $p_t^\perp q_t^\perp = q_t^\perp$ for (Lebesgue) almost every $t \in [0, \infty)$; i.e., $q_t^\perp \le p_t^\perp$ for almost every $t \in [0, \infty)$, for each $\tau \in \Lambda$. If Λ is countable, it follows that $\sup_\Lambda q_t^\perp \le p_t^\perp$ for almost every $t \in [0, \infty)$. Hence $\sup \Lambda \le \sigma$.

But, on the other hand, $\tau \le \sup \Lambda$ for each $\tau \in \Lambda$ and so $M_\tau \le M_{\sup\Lambda}$ for each $\tau \in \Lambda$. Hence $\sup \mathscr{F} = \sup\{M_\tau : \tau \in \Lambda\} \le M_{\sup\Lambda}$. We conclude that the equality $\sup \mathscr{F} = M_q = M_{\sup\Lambda}$ holds.

The argument for the infima is similar. □

Remark. In the course of the above proof, we have proved a converse to the Optimal Stopping Theorem, Theorem 2.9; namely, that if $\sigma = (p_t)$, $\tau = (q_t)$, and $M_\tau \le M_\sigma$, then $p_t \le q_t$ for (Lebesgue) almost all $t \in [0, \infty]$.

4. Stochastic integration

Stochastic integration with respect to an arbitrary L^2-martingale was discussed in [7]. We wish to present, in this section, a slightly different treatment of this topic. We shall define stochastic integration in a manner entirely analogous to the approahc taken in classical probability via the natural increasing process of the Doob–Meyer decomposition of the (modulus) square of the L^2-martingale. We shall restrict our attention to centered L^2-martingales; that is, to those orthogonal to Ω (equivalently, those vanishing at 0) and denote the set of these by $\mathscr{M}_0^2$.

We recall [7, 8] that for any $\zeta = (0, \xi, \eta) \in \mathscr{M}_0^2$ one can define its "square" $|\zeta|^2$, which is an L^1-process (in fact, a submartingale) and which can be written as the sum of an L^1-martingale and an increasing L^1-process $(\langle \zeta, \zeta \rangle_t)$ given by

$$\langle \zeta, \zeta \rangle_t(\cdot) = \int_0^t (\cdot J\xi(s), J\xi(s))(1 - \rho(s))\, ds + \int_0^t (\cdot J\eta(s), J\eta(s))\rho(s)\, ds .$$

Definition 4.1. Let g be an elementary $\mathfrak{A}$-valued process; that is, g has the form

$$g(s) = x\chi_{[a.b)}(s)$$

for some $0 \le a < b$ and $x \in \mathfrak{A}_a$. The *stochastic integral* of g with respect to $\zeta \in \mathscr{M}_0^2$ is

$$\int_0^\infty g\, d\zeta = x(\zeta_b - \zeta_a) .$$

The stochastic integral for simple g (i.e., when g is a finite linear combination of elementary $\mathfrak{A}$-valued processes) is defined by linearity.

Proposition 4.2. *Let* $\zeta = (0, \xi, \eta) \in \mathscr{M}_0^2$. *For any simple* $\mathfrak{A}$*-valued process* g, *we have*

$$\int_0^\infty g\, d\zeta = \int_0^\infty db_s^* \beta(g(s))\xi(s) + \int_0^\infty db_s \beta(g(s))\eta(s)$$

and

$$\begin{aligned}\left\| \int_0^\infty g\, d\zeta \right\|_2^2 &= \int_0^\infty (\beta(|g(s)|^2)\xi(s), \xi(s))(1 - \rho(s))\, ds \\ &\quad + \int_0^\infty (\beta(|g(s)|^2)\eta(s), \eta(s))\rho(s)\, ds \\ &= \int_0^\infty \beta(|g(s)|^2) d\langle \tilde{\zeta}, \tilde{\zeta} \rangle(\mathbf{1})\end{aligned}$$

where $\tilde{\zeta} = (0, J\xi, J\eta)$. *(Note that* $L^\infty(\mathfrak{A}) = \mathfrak{A}$ *acts on* $L^1(\mathfrak{A}) \simeq \mathfrak{A}_*$ *by the usual left action* $(z\phi)(\cdot) = \phi(\cdot z)$, $z \in \mathfrak{A}$, $\phi \in \mathfrak{A}_*$.)

Proof. This follows readily from the construction of the stochastic integrals with respect to b_s^* and b_s, the isometry relations, and the formula for the increasing process in the Doob–Meyer decomposition of $|\zeta|^2$. See [7] for further details. □

It follows from this proposition that the stochastic integral of $\beta(|g|^2)$ with respect to $\langle \tilde{\zeta}, \tilde{\zeta} \rangle$, when evaluated at $\mathbf{1}$, gives rise to a quadratic form on the simple processes. Thus, following the classical approach (see [10], for example) we make the following definition.

Definition 4.3. For a simple $\mathfrak{A}$-valued processes f, g and for a given centered L^2-martingale $\zeta \in \mathscr{M}_0^2$, we define the *sesquilinear form*

$$\langle f, g\rangle = \int_0^\infty \beta(f^* g)d\langle \tilde{\zeta}, \tilde{\zeta}\rangle(\mathbf{1}).$$

The kernel of this form is a linear subspace of the linear space of simple $\mathfrak{A}$-valued processes, and so we factor by these—or, equivalently, identify processes whose difference lies in the kernel. With this identification, the set of simple processes equipped with this sesquilinear form becomes a preHilbert space and the stochastic integral with respect to ζ becomes an isometry of this space into $L^2(\mathfrak{A})$. We denote by $L^1(\zeta)$ the completion of the space of (equivalence classes of) simple processes and will continue to write $\int_0^\infty f\,d\zeta$ for the extension of the stochastic integral with respect to ζ to all of $L^1(\zeta)$. Of course, the integral with respect to ζ is an isometry of $L^1(\zeta)$ into $L^2(\mathfrak{A})$, by construction, and this characterizes the integral *as a map*, given its value on the elementary integrands. The *process* obtained by stochastic integration; namely, $\{\int_0^t f\,d\zeta : t \in \mathbf{R}^+\}$ can be characterized as in [7].

We shall not address the question of identifying $L^1(\zeta)$ as a space of *concrete* processes, but refer to [7] where there are results which go some way in that direction.

In the next section, we shall investigate the range of the stochastic integral.

5. The time algebra

In this section, we shall show that the range of the stochastic integral is given by the range of the algebra of time projections.

Definition 5.1. Let $\mathscr{T}$ denote the von Neumann algebra in $\mathscr{H}$ generated by the time projections; $\mathscr{T} = \{M_\tau : \tau \text{ a random time}\}''$. Let $\mathscr{T}_0$ denote the $*$-algebra generated by finite products of time projections.

Proposition 5.2. *Let f be an $\mathfrak{A}$-valued process which is the (Lebesgue) almost everywhere limit in the strong operator topology of a uniformly bounded sequence of simple $\mathfrak{A}$-valued processes (this is the class $\mathscr{P}([0, \infty))$ of [7]). Then $f \in L^1(\zeta)$ for any $\zeta \in \mathscr{M}_0^2$.*

Proof. This is an application of the martingale representation theorem and dominated convergence; see [7]. □

Corollary 5.3. *Let f be as above. Then the map $\zeta \mapsto \int_0^\infty f\,d\zeta$ defines a bounded linear operator on $\mathscr{H}$ where $\zeta \in \mathscr{H}$ has been identified with the L^2-bounded martingale $\zeta_t = P_t\zeta$, $t \in [0, \infty)$.*

Proof. If $\zeta = (\alpha, \xi, \eta)$, then $\int_0^\infty f\, d\zeta = (0, \beta(f)\xi, \beta(f)\eta)$ by [7]. Hence, by the isometry relations,

$$\left\| \int_0^\infty f\, d\zeta \right\|_2^2 = \int_0^\infty \|\beta(f(s))\xi(s)\|^2 (1 - \rho(s))\, ds + \int_0^\infty \|\beta(f(s))\eta(s)\|^2 \rho(s)\, ds$$
$$\leq \left(\sup_s \|f(s)\|_\infty \right) \|\zeta\|_2^2 . \quad \square$$

Corollary 5.4. *Let $T \in \mathscr{T}_0$. Then there is a process f, as above, such that*

$$T\zeta = P_0\zeta + \int_0^\infty f\, d\zeta$$

for any $\zeta \in \mathscr{H}$ (again, (ζ_t) is the L^2-martingale $(P_t\zeta)$).

Proof. by linearity, it is enough to suppose that T is a product of time projections, $T = M_{\tau_1} \cdots M_{\tau_n}$, where $\tau_j = (q_t^{(j)})$, $j = 1, \ldots, n$, are random times. Then if $\zeta = (\alpha, \xi, \eta)$, we have

$$T\zeta = \alpha\Omega + \int_0^\infty db_s^* \beta(q_s^{(1)\perp} \cdots q_s^{(n)\perp})\xi(s) + \int_0^\infty db_s \beta(q_s^{(1)\perp} \cdots q_s^{(n)\perp})\eta(s).$$

Now, $\alpha\Omega = P_0\zeta$ and, by Proposition 2.3 (or directly), we see that $f(\cdot) = \beta(q_\cdot^{(1)\perp} \cdots q_\cdot^{(n)\perp})$ is the (Lebesgue) almost everywhere strong operator limit of a uniformly bounded sequence of simple $\mathfrak{A}$-valued processes, and so the result follows. $\square$

Remark. The following theorem is the counterpart of the result from classical probability which states that the range of the stochastic integral with respect to an L^2-martingale is generated by all bounded stopping of that martingale.

Theorem 5.5. *For each $\zeta \in \mathscr{H}$, the closed subspace $R(\zeta) = \{\int_0^\infty f\, d\zeta : f \in L^1(\zeta)\}$ is equal to the closure of the subspace $\mathscr{T}\zeta = \{T\zeta : T \in \mathscr{T}\}$.*

Proof. We have seen that $\mathscr{T}_0\zeta \subseteq R(\zeta)$, and hence $\overline{\mathscr{T}_0\zeta} \subseteq R(\zeta)$. But $\overline{\mathscr{T}_0\zeta} = \overline{\mathscr{T}\zeta}$, since $\mathscr{T}$ is the strong closure of $\mathscr{T}_0$, and hence $\overline{\mathscr{T}\zeta} \subseteq R(\zeta)$.

To show that $R(\zeta) \subseteq \overline{\mathscr{T}\zeta}$, let h be an elementary process of the form

$$h(s) = q\chi_{[r,t)}(s),$$

with $0 < r < t$ and where q is a projection in $\mathfrak{A}_r$.

Define

$$q_s' = \begin{cases} 0, & s < r \\ q, & r \leq s < \infty \\ \mathbf{1}, & s = \infty \end{cases}, \qquad q_s'' = \begin{cases} 0, & s < t \\ q, & t \leq s < \infty \\ \mathbf{1}, & s = \infty . \end{cases}$$

Then $\tau' = (q_s')$, $\tau'' = (q_s'')$ are random times, and

$$(M_\tau' - M_\tau'')\zeta = \int_0^\infty h\, d\zeta .$$

Hence $\int_0^\infty h\, d\zeta \in \mathscr{T}\zeta$.

Now let $g(s) = x\chi_{[r,t)}(s)$, $0 < r < t$, be an elementary process with $x \in \mathfrak{A}_r$. Then x is a linear combination of four positive operators, each of which is a norm limit of a bounded sequence (y_n) of finite linear combinations of projections belonging to $\mathfrak{A}_r$ (by the spectral theorem).

By the analysis above, it follows that each $\int_0^\infty y_n\chi_{[r,t)}\,d\zeta$ belongs to $\mathscr{T}\zeta$, and so $\int_0^\infty g\,d\zeta \in \overline{\mathscr{T}\zeta}$. By linearity, it follows that $\int_0^\infty f\,d\zeta \in \overline{\mathscr{T}\zeta}$ for any simple $\mathfrak{A}$-valued process f and hence, by continuity, $R(\zeta) \subseteq \overline{\mathscr{T}(\zeta)}$. □

Remark. We would like to reiterate at this point that all of the results of §3, §4, and §5 have analogues within the quantum stochastic theory of the CCR [3]. As noted in [7], given the setup and the stochastic integral representation theorem [15], the translation from the CAR to the CCR is quite routine (and amounts to little more than replacing β, the parity automorphism, by the idenitty automorphism, and replacing $\rho(s)$ by $\gamma(s)$ and $1-\rho(s)$ by $1+\gamma(s)$ where γ is the nonnegative function on $\mathbf{R}^+$ which determines the two-point functions which itself determines the gauge-invariant quasi-free representation of the CCR under consideration. (In [3], [7], [15], this function was denoted by τ rather than γ. We have suggested γ here to avoid possible notational confusion with random times.)

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