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THE CESÀRO OPERATOR IS BOUNDED ON H^1

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ABSTRACT. The purpose of this note is to give a direct proof that the Cesàro operator is bounded on the Hardy space H^1.

1. INTRODUCTION

On the Hardy spaces H^p of the unit disc, the Cesàro operator is

$$\mathscr{C}(f)(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n, \tag{1}$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is in H^p. It is known that $\mathscr{C}$ is bounded on H^p for $1 \leq p < \infty$. For $1 < p < \infty$ this can be obtained by using the Hardy result [1] concerning trigonometric series and M. Riesz's theorem, but this proof does not cover the case $p = 1$. The boundedness of $\mathscr{C}$ on H^1 was obtained as a byproduct in [3], where, for the purpose of finding the norm of $\mathscr{C}$ on H^p, a related strongly continuous semigroup of weighted composition operators was studied. The purpose of this note is to give a direct proof of the boundedness of $\mathscr{C}$ on H^1, avoiding the semigroups.

We use the usual notation $M_p(r, f)$ for the integral means on $|z| = r$ of an analytic f, and will make use of the following Hardy–Littlewood result which we state as a lemma.

Lemma [2, p. 412]. *If $f \in H^1$ and $q > 1$ then*

$$\int_0^1 M_q(r, f)(1-r)^{-1/q} dr \leq k\|f\|_1$$

and the constant k depends only on q.

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2. The proof

A computation shows that (1) can be written

$$\begin{aligned}\mathscr{C}(f)(z) &= \frac{1}{z}\int_0^z f(\zeta)\frac{1}{1-\zeta}\,d\zeta \\ &= \int_0^1 f(tz)\frac{1}{1-tz}\,dt.\end{aligned}$$

For $0 < r < 1$ and $f \in H^1$, we have

$$\begin{aligned}M_1(r, \mathscr{C}(f)) &= \frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^1 f(rte^{i\theta})\frac{1}{1-rte^{i\theta}}\,dt\right|\,d\theta \\ &\le \int_0^1 \frac{1}{2\pi}\int_0^{2\pi}|f(rte^{i\theta})|\frac{1}{|1-rte^{i\theta}|}\,d\theta\,dt \\ &\le \int_0^1\left(\frac{1}{2\pi}\int_0^{2\pi}|f(rte^{i\theta})|^2\,d\theta\right)^{1/2}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{1-rte^{i\theta}}\right|^2\,d\theta\right)^{1/2}\,dt \\ &= \int_0^1 M_2(rt, f)\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-r^2t^2}P(rt, \theta)\,d\theta\right)^{1/2}\,dt,\end{aligned}$$

where Hölder's inequality was used in the third step and $P(r, \theta)$ is the Poisson kernel. Since $(1/2\pi)\int_0^{2/\pi} P(r, \theta)\,d\theta = 1$, the last integral is dominated by

$$\int_0^1 M_2(rt, f)(1-r^2t^2)^{-1/2}\,dt \le \int_0^1 M_2(t, f)(1-t)^{-1/2}\,dt \le k\|f\|_1,$$

by applying the Lemma with $q = 2$ in the last step. This finishes the proof.

It is interesting to see, upon close examination of the argument, that the proof cannot be adapted for other values $0 < p < 1$ or $1 < p < \infty$.

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