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EVERY ZERO-DIMENSIONAL SPACE IS CELL SOLUBLE

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ABSTRACT. In his study of the question of representing a space as a retract of a homogeneous space, Arhangel'skii introduced an interesting topological property called *cell solubility*. He raised the following problem: Is every zero-dimensional compact space cell soluble? We will give an affirmative answer to this problem.

1. INTRODUCTION

A topological space X is called homogeneous if for arbitrary points $x, y \in X$ there exists a homeomorphism f from X onto itself such that $f(x) = y$. Since topological groups are homogeneous and any power S^κ of the circle S is a compact topological group, every Tychonoff space is embedded in some homogeneous compact T_2-space. On the other hand, Motorov (see [1] or [2]) showed that there exists a metrizable compact space that is not a retract of any homogeneous compact T_2-space. When is a compact T_2-space X a retract of some homogeneous compact T_2-space? Concerning this problem, Arhangel'skii found an interesting topological property that every retract of an arbitrary homogeneous compact T_2-space possesses.

Let $q = (Y, Z, \mathscr{E})$ be a triple, where Y is a topological space, Z a subspace of Y, and $\mathscr{E}$ a nonempty family of subsets of Y. Let X be an arbitrary topological space. A closed subset P of X is said to be *q-saturated* if for any continuous map $f: Y \to X$ such that $f(L) \cap P \neq \varnothing$ for all $L \in \mathscr{E}$ we have $f(Z) \subset P$. For an arbitrary point $x \in X$ we denote by $F_q(X)$ the intersection of all q-saturated subsets containing x. The family $\{F_q(x): x \in X\}$ is called the *cellularity* induced by the triple q. The sets $F_q(x)$ are called the *terms* of the cellularity. The cellularity $\{F_q(x): x \in X\}$ is called *disjoint* if for all $x, y \in X$ either $F_q(x) = F_q(y)$ or $F_q(x) \cap F_q(y) = \varnothing$. A topological space X is called *cell soluble* if for any triple q (as above), its induced cellularity is disjoint, provided that at least one of its terms is compact (see [1] or [2]).

Arhangel'skii proved that every retract of an arbitrary homogeneous compact T_2-space is cell soluble. Moreover, he raised the following problem.

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Problem. Is every zero-dimensional compact T_2-space cell soluble?

In this note we will give an affirmative answer to this problem.

2. Conclusion

We recall that a T_1-space X is called zero-dimensional if X has an open base consisting of clopen subsets.

Theorem. *Every zero-dimensional topological space is cell soluble.*

Proof. Let X be a zero-dimensional topological space. Let $q = (Y, Z, \mathscr{E})$ be an arbitrary triple, where Y is a topological space, Z a subspace of Y, and $\mathscr{E}$ a nonempty family of subsets of Y. If $Z = \varnothing$ and $f\colon Y \to X$ is continuous, then $F_q(x) = \{x\}$ for any $x \in X$ since $f(\varnothing) = \varnothing \subset \{x\}$. Hence we can assume that $Z \neq \varnothing$.

Suppose that there exists a clopen subset G of Y such that $G \cap Z \neq \varnothing$ and $L \backslash G \neq \varnothing$ for every $L \in \mathscr{E}$. Then we can show that $F_q(x) = X$ for any $x \in X$. In fact, if a subset P of X satisfies $P \neq X$, then there exists a continuous map $f\colon Y \to X$ such that $f(G) \subset X \backslash P$ and $f(Y \backslash G) \subset P$. This shows that P is not q-saturated.

The case remaining is the following: For any clopen subset G of Y, if $G \cap Z \neq \varnothing$, then there exists some member L of $\mathscr{E}$ such that $L \subset G$. In this case, it will be proved that $F_q(x) = \{x\}$ for any $x \in X$. Let $f\colon Y \to X$ be a continuous map such that $f(L) \cap \{x\} \neq \varnothing$ for any $L \in \mathscr{E}$. Then it suffices to show that $f(Z) = \{x\}$. If $f(Z) \neq \{x\}$, then there exists a clopen set U of X such that $U \cap f(Z) \neq \varnothing$ and $x \notin U$ are satisfied. But this is a contradiction since the clopen subset $f^{-1}(U)$ satisfies $f^{-1}(U) \cap Z \neq \varnothing$ and $L \backslash f^{-1}(U) \neq \varnothing$ for any $L \in \mathscr{E}$. This completes the proof.

Remark 1. From our result it follows that another problem of Arhangel'skii [2] is solved simultaneously.

Problem. Is it true that the space $\beta N \backslash N$ (respectively, the growth of the space $\beta(\tau)$) is cell soluble?

Remark 2. Van Douwen [3] posed the following very interesting problem.

Problem. Is a compact T_2-space nonhomogeneous if it can be mapped continuously onto βN?

It is well known that this problem is equivalent to the following:

Problem. Is βN the retract of some homogeneous compact T_2-space?

Our result shows that a technique like that of Motorov is not appropriate for solving the last problem.

References

1. A. V. Arhangel'skii, *Cell structures and homogeneity*, Mat. Zametki **37** (1985), 580–586; Math. Notes **37** (1985), 321–324.

2. ——, *Topological homogeneity. Topological groups and their continuous images*, Uspekhi Mat. Nauk **2** (1987), 69–105; Russian Math. Surveys **2** (1987), 83–131.

3. E. K. van Douwen, *Non-homogeneity of products of preimages and π-weight*, Proc. Amer. Math. Soc. **69** (1978), 183–192.

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