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THE CONSTRUCTION OF GLOBAL ATTRACTORS

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ABSTRACT. The purpose of this note is to show that every inverse limit space of an interval mapping can be realized as a global attractor for a homeomorphism of the plane.

The purpose of this note is to describe a simple method for the construction of an abundance of global attractors. In order to facilitate understanding, we describe this method in the plane.

Suppose that I is an interval and that $f: I \to I$ is continuous. Let (I, f) be the inverse limit space $\{(x_0, x_1, \ldots) | x_i \in I \text{ and } f(x_{i+1}) = x_i\}$, with metric $d((x_0, x_1, \ldots), (y_0, y_1, \ldots)) = \sum_{i=0}^{\infty} |x_i - y_i|/2^i$. We will show that (I, f) can be topologically realized as a global attractor in the plane. These inverse limit spaces are examples of what Bing has called "snakelike continua", see [Bi] or [Wa]. Furthermore, the dynamics on (I, f) can be understood in terms of the dynamics of f, see [B-M1; B-M2; B-M3]. In [Wi], Williams discusses inverse limits of branched 1-manifolds as "generalized solenoids", which have attracting neighborhoods.

The idea is a simple one. Imagine that D is a disk and that $I \subset \operatorname{int} D$. We construct a map $H: D \to D$ such that (1) $H(I) = I$, and $H|I = f$; (2) $H|\operatorname{Bdry}(D) = \operatorname{id}$; (3) if $x \in \operatorname{int} D$ there is a positive integer n, such that $H^n(x) \in I$; and (4) H is uniformly approximated by homeomorphisms. Then let X be the inverse limit of D with bonding map H. Using (4), it follows from a theorem of M. Brown [Br] that X is a topological disk. Using conditions (1) and (3) we see that (I, f) is embedded in $X = (D, H)$, and if $x \in \operatorname{int} X$, then $d((\hat{H})^n(x), (I, f)) \to 0$. Here $\hat{H}$ is the homeomorphism on X induced by H. Furthermore, the homeomorphism $\hat{H}|(I, f)$ is just the homeomorphism $\hat{f}: (I, f) \to (I, f)$, induced by f.

Definitions. If Z is a compact metric space, and $g: Z \to Z$ is continuous, the *inverse limit space* (Z, g) is $\{(z_0, z_1, \ldots) | z_i \in Z \text{ and } f(z_{i+1}) = z_i\}$ with the metric $\rho((z_0, z_1, \ldots), (y_0, y_1, \ldots)) = \sum_{i=0}^{\infty} d(z_i, y_i)/2^i$. The *in-*

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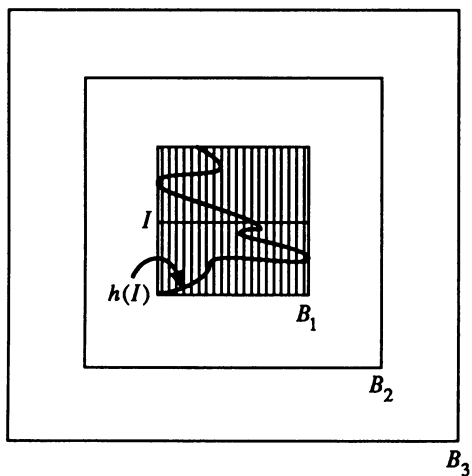
duced homeomorphism $\hat{g}: (Z, g) \twoheadrightarrow (Z, g)$ is given by $\hat{g}((z_0, z_1, \ldots)) = (g(z_0), z_0, z_1, \ldots)$.

If A is a subset of the plane E^2, the statement that A is a *global attractor* means that there is a homeomorphism $h: E^2 \twoheadrightarrow E^2$ such that (1) $h(A) = A$; (2) if $x \in E^2$ then $d(h^n(x), A) \to 0$ as $n \to \infty$ and; (3) if U is open and $A \subset U$, then there is an open set V and a positive integer N such that $A \subset V \subset U$ and if $n > N$, then $h^n(V) \subset U$.

Construction of the examples. For $i = 1, 2, 3$, let $B_i \subset E^2$ be $\{(x, y) | -i \leq x \leq i \text{ and } -i \leq y \leq i\}$. Let I be the interval $\{(t, 0) | -1 \leq t \leq 1\}$ and suppose that $f: I \to I$ is continuous.

Now let $h: B_3 \to B_3$ be a homeomorphism such that (1) $h|B_3 - B_2 = \mathrm{id}$; (2) if $(t, 0) \in I$ then $h((t, 0)) = (f(t), t)$. This last condition insures that h, followed by vertical projection onto I, is f. See the diagram.



We now construct a continuous function $G: B_3 \times [0, 1] \to B_3$. Denoting $G|B_3 \times \{t\}$ by G_t, we will have the following properties:

(1) $G_0 = \mathrm{id}$;
(2) G_t is a homeomorphism if $0 \leq t < 1$;
(3) for each t, $G_t|\mathrm{Bdry}(B_3) = \mathrm{id}$;
(4) if $(t, 0) \in I$, then $\{(t, s) | -1 \leq s \leq 1\} \subset G_1^{-1}((t, 0))$;
(5) $G_1(B_2) = B_1$;
(6) if $x \in \mathrm{int}\, B_3$, there is an integer n such that $G_1^n(x) \in I$.

Roughly speaking, B_2 is gradually squeezed down to B_1 while the vertical intervals in B_1 are shrunk down to points in I.

Now, let $H = G_1 \circ h$ and let $X = (B_3, H)$ be the inverse limit space of B_3 with bonding map H. From Condition (2), it follows that H is uniformly approximated by homeomorphisms. Using [Br, Theorem 4], it follows that X is a topological disk. Let $A = \{(x_0, x_1, \dots) | (x_0, x_1, \dots) \in X \text{ and } x_i \in I\}$. Then $A \subset X$, and it follows from (5), (6), and the fact that $h = \text{id}$ on $B_3 - B_2$, that A is a global attractor for $\operatorname{int} X$ under $\hat{H}$.

Now suppose that $(t, 0) \in I$. Then $H(t, 0) = G_1((f(t), t)) = (f(t), 0)$, by (4). From this it follows that A is homeomorphic with (I, f). Notice that if $(x_0, x_1, \dots) \in A$, then $\hat{H}((x_0, x_1, \dots)) = (H(x_0), x_0, x_1, \dots) = (f(x_0), x_0, \dots) = \hat{f}((x_0, x_1, \dots))$. so the homeomorphism induced on A by $\hat{H}$ is just $\hat{f}$.

Remarks. Notice that if $p \in \operatorname{int} X$, then there is a point $q \in A$ such that $d((\hat{H})^n(p), (\hat{H})^n(q)) \to 0$. Also it is clear that this construction, and elaborations of it, can be carried out in much greater generality. We will discuss these results elsewhere.

REFERENCES

[B-M1] M. Barge and J. Martin, *Chaos, periodicity, and snakelike continua*, Trans. Amer. Math. Soc. **289** (1985), 355–363.

[B-M2] ——, *Dense orbits on the intervals*, Michigan Math. J. **34** (1987), 3–11.

[B-M3] ——, *Dense periodicity on the interval*, Proc. Amer. Math. Soc. **94** (1985), 731–736.

[Bi] R. H. Bing, *Snakelike continua*, Duke Math. J. **18** (1951), 653–663.

[Br] M. Brown, *Some applications of an approximation theorem for inverse limits*, Proc. Amer. Math. Soc. **11** (1960), 478–483.

[Wa] W. Watkins, *Homeomorphic classification of certain inverse limit spaces with open bonding maps*, Pacific J. Math. **103** (1982), 589–601.

[Wi] R. F. Williams, *One-dimensional non-wandering sets*, Topology **6** (1967), 473–483.

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