THE CONSTRUCTION OF GLOBAL ATTRACTORS

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Abstract. The purpose of this note is to show that every inverse limit space of an interval mapping can be realized as a global attractor for a homeomorphism of the plane.

The purpose of this note is to describe a simple method for the construction of an abundance of global attractors. In order to facilitate understanding, we describe this method in the plane.

Suppose that \( I \) is an interval and that \( f: I \to I \) is continuous. Let \( (I, f) \) be the inverse limit space \( \{(x_0, x_1, \ldots) | x_i \in I \text{ and } f(x_{i+1}) = x_i \} \), with metric \( d((x_0, x_1, \ldots), (y_0, y_1, \ldots)) = \sum_{i=0}^{\infty} |x_i - y_i|/2^i \). We will show that \( (I, f) \) can be topologically realized as a global attractor in the plane. These inverse limit spaces are examples of what Bing has called “snakelike continua”, see [Bi] or [Wa]. Furthermore, the dynamics on \( (I, f) \) can be understood in terms of the dynamics of \( f \), see [B-M1; B-M2; B-M3]. In [Wi], Williams discusses inverse limits of branched 1-manifolds as “generalized solenoids”, which have attracting neighborhoods.

The idea is a simple one. Imagine that \( D \) is a disk and that \( I \subset \text{int} D \). We construct a map \( H: D \to D \) such that (1) \( H(I) = I \), and \( H|I = f \); (2) \( H|\text{Bdry}(D) = \text{id} \); (3) if \( x \in \text{int} D \) there is a positive integer \( n \), such that \( H^n(x) \in I \); and (4) \( H \) is uniformly approximated by homeomorphisms. Then let \( X \) be the inverse limit of \( D \) with bonding map \( H \). Using (4), it follows from a theorem of M. Brown [Br] that \( X \) is a topological disk. Using conditions (1) and (3) we see that \( (I, f) \) is embedded in \( X = (D, H) \), and if \( x \in \text{int} X \), then \( d((\tilde{H})^n(x), (I, f)) \to 0 \). Here \( \tilde{H} \) is the homeomorphism on \( X \) induced by \( H \). Furthermore, the homeomorphism \( \tilde{H}|(I, f) \) is just the homeomorphism \( \tilde{f}: (I, f) \to (I, f) \), induced by \( f \).

Definitions. If \( Z \) is a compact metric space, and \( g: Z \to Z \) is continuous, the inverse limit space \( (Z, g) \) is \( \{(z_0, z_1, \ldots) | z_i \in Z \text{ and } g(z_{i+1}) = z_i \} \) with the metric \( \rho((z_0, z_1, \ldots), (y_0, y_1, \ldots)) = \sum_{i=0}^{\infty} d(z_i, y_i)/2^i \).
duced homeomorphism \( g : (Z, g) \to (Z, g) \) is given by \( g((z_0, z_1, \ldots)) = (g(z_0), z_0, z_1, \ldots) \).

If \( A \) is a subset of the plane \( E^2 \), the statement that \( A \) is a global attractor means that there is a homeomorphism \( h : E^2 \to E^2 \) such that (1) \( h(A) = A \); (2) if \( x \in E^2 \) then \( d(h^n(x), A) \to 0 \) as \( n \to \infty \) and; (3) if \( U \) is open and \( A \subset U \), then there is an open set \( V \) and a positive integer \( N \) such that \( A \subset V \subset U \) and if \( n > N \), then \( h^n(V) \subset U \).

**Construction of the examples.** For \( i = 1, 2, 3 \), let \( B_i \subset E^2 \) be \( \{(x, y) \mid -t < x < i \text{ and } -i < y < i\} \). Let \( I \) be the interval \( \{(t, 0) \mid -1 < t < 1\} \) and suppose that \( f : I \to I \) is continuous.

Now let \( h : B_3 \to B_3 \) be a homeomorphism such that (1) \( h|B_3 - B_2 = \text{id} \); (2) if \( (t, 0) \in I \) then \( h((t, 0)) = (f(t), t) \). This last condition insures that \( h \), followed by vertical projection onto \( I \), is \( f \). See the diagram.

We now construct a continuous function \( G : B_3 \times [0, 1] \to B_3 \). Denoting \( G|B_3 \times \{t\} \) by \( G_t \), we will have the following properties:

- (1) \( G_0 = \text{id} \);
- (2) \( G_t \) is a homeomorphism if \( 0 \leq t < 1 \);
- (3) for each \( t \), \( G_t|\text{Bdry}(B_3) = \text{id} \);
- (4) if \( (t, 0) \in I \), then \( \{(t, s) \mid -1 \leq s \leq 1\} \subset G^{-1}_i((t, 0)) \);
- (5) \( G_1(B_2) = B_1 \);
- (6) if \( x \in \text{int} B_3 \), there is an integer \( n \) such that \( G^n_i(x) \in I \).

Roughly speaking, \( B_2 \) is gradually squeezed down to \( B_1 \) while the vertical
intervals in \( B_1 \) are shrunk down to points in \( I \).

Now, let \( H = G_1 \circ h \) and let \( X = (B_3, H) \) be the inverse limit space of \( B_3 \) with bonding map \( H \). From Condition (2), it follows that \( H \) is uniformly approximated by homeomorphisms. Using [Br, Theorem 4], it follows that \( X \) is a topological disk. Let \( A = \{(x_0, x_1, \ldots) | (x_0, x_1, \ldots) \in X \text{ and } x_i \in I \} \). Then \( A \subset X \), and it follows from (5), (6), and the fact that \( h = \text{id} \) on \( B_3 - B_2 \), that \( A \) is a global attractor for \( \text{int } X \) under \( H \).

Now suppose that \((t, 0) \in I \). Then \( H(t, 0) = G_1((f(t), t)) = (f(t), 0) \), by (4). From this it follows that \( A \) is homeomorphic with \((I, f)\). Notice that if \((x_0, x_1, \ldots) \in A \), then \( H((x_0, x_1, \ldots)) = (H(x_0), x_0, x_1, \ldots) = (f(x_0), x_0, \ldots) = f((x_0, x_1, \ldots)) \). so the homeomorphism induced on \( A \) by \( H \) is just \( f \).

Remarks. Notice that if \( p \in \text{int } X \), then there is a point \( q \in A \) such that \( d((H)^n(p), (H)^n(q)) \to 0 \). Also it is clear that this construction, and elaborations of it, can be carried out in much greater generality. We will discuss these results elsewhere.

References