PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 110, Number 2, October 1990

# AN ISOMORPHISM THEOREM FOR COMMUTATIVE MODULAR GROUP ALGEBRAS

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(Communicated by Warren J. Wong)

ABSTRACT. For each positive integer $n$ and limit ordinal $\mu$, a new class of abelian $p$-groups, called $A_n(\mu)$-groups, are introduced. These groups are shown to be uniquely determined up to isomorphism by numerical invariants which include, but are not restricted to, their Ulm-Kaplansky invariants. As an application of this uniqueness theorem, we prove an isomorphism result for group algebras: Let $H$ be an $A_n(\mu)$-group and $F$ a field of characteristic $p$. It is shown that if $K$ is a group such that the group algebras $FH$ and $FK$ are $F$-isomorphic, then $H$ and $K$ are isomorphic.

## 1. INTRODUCTION

Suppose $F$ is a field of characteristic $p > 0$, and let $H$ be an abelian $p$-group. If $K$ is a group such that the group algebras $FH$ and $FK$ are $F$-isomorphic, the question of whether $H \cong K$ arises. It has been known for some time that the Ulm-Kaplansky invariants of $H$ and $K$ must be equal [1, 6]. Thus, if $H$ is countable it follows that $H \cong K$ by the well-known Ulm's Theorem for countable abelian $p$-groups.

Several generalizations of this basic result have been made over the last twenty years. In 1969, Berman and Mollov [2] showed that if $H$ is an arbitrary direct sum of cyclic $p$-groups, then $K$ must also be such a group, hence $H \cong K$. Later, this result was extended [7] to the case where $H$ is totally projective of length less than the first uncountable ordinal. More recently, the cases where $H$ is totally projective [8], an $N$-group [9, 10], and an elementary $A$-group [10] have been shown to imply $H \cong K$.

In this paper we further extend these known isomorphism results. In §2, for each positive integer $n$ and limit ordinal $\mu$, we define a class of $p$-groups, which we call $A_n(\mu)$-groups. In the case $n = 1$ these are the $\mu$-elementary $A$-groups of Hill [4]. Moreover, the class $A_n(\mu)$ consists entirely of totally projective groups if and only if $\mu$ is cofinal with $\omega_0$ (Lemma 1). Thus, in case

Received by the editors March 13, 1989 and, in revised form, January 19, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20K10; Secondary 20C07.
*Key words and phrases.* Modular group algebras, invariants for abelian $p$-groups.

$n \geq 2$ and $\mu$ is not cofinal with $\omega_0$, new classes of $p$-groups are obtained (see the example following Theorem 2).

In §3 we introduce cardinal invariants for $A_n(\mu)$-groups and prove a result (Theorem 2) which may be of independent interest; namely, two $A_n(\mu)$-groups are isomorphic if and only if their associated invariants are equal. We remark that these invariants include, but are not restricted to, the Ulm-Kaplansky invariants. Finally, in the last section, we apply the uniqueness theorem of §3 to extend the group algebra isomorphism results discussed above to the case where $H$ is an $A_n(\mu)$-group, for any positive integer $n$ and limit ordinal $\mu$.

Throughout, $p$ is an arbitrary fixed prime and all groups considered are $p$-primary abelian groups, written multiplicatively. Abelian group terminology not explicitly defined herein is in agreement with Fuchs [3].

## 2. A CLASS OF $p$-GROUPS

If $G$ is a $p$-group and $\sigma$ is an ordinal, we write $G^\sigma$ for the subgroup of $G$ consisting of the elements of $G$ of heights $\geq \sigma$. If $G$ has limit length $\mu$, an isotype subgroup $H$ of $G$ is called an *almost balanced* subgroup of $G$ provided that $(G/H)^\sigma = G^\sigma H/H$ for every $\sigma < \mu$.

Fix a limit ordinal $\mu$. For each positive integer $n$ we define a class of $p$-groups $A_n(\mu)$ as follows. Call $H$ an *$A_1(\mu)$-group* if there exists a totally projective group $G$ of length $\mu$ such that $H$ is almost balanced in $G$ and $G/H$ is totally projective. For $n \geq 2$, call $H$ an *$A_n(\mu)$-group* if there exists a totally projective group $G$ of length $\mu$ satisfying the following conditions:

(a) $H$ is an almost balanced subgroup of $G$.

(b) $(G/H)^\mu$ is an $A_{n-1}(\mu)$-group.

(c) $(G/H)/(G/H)^\mu$ is totally projective.

If $G$ and $H$ satisfy the conditions set forth above, for some positive integer $n$, we call $(H, G)$ an *$A_n(\mu)$-pair.*

Several comments are in order. First, we do not require totally projective groups to be reduced. If desired, it is no great loss to consider only reduced $A_n(\mu)$-groups; however, we shall not do this. We also remark that our $A_1(\mu)$-groups are the $\mu$-elementary $A$-groups as defined in [4]. Thus, our introduction of the classes $A_n(\mu)$ may be viewed as a modest attempt to extend P. Hill's theory of $A$-groups. Indeed, our results for $A_n(\mu)$-groups are direct extensions of known results for $\mu$-elementary $A$-groups. Since the class of $\mu$-elementary $A$-groups consists entirely of totally projective groups if and only if $\mu$ is cofinal with $\omega_0$ [4, Theorem 1], a routine induction yields our first result.

**Lemma 1.** *For each positive integer $n$, the class $A_n(\mu)$ consists entirely of totally projective groups if and only if $\mu$ is cofinal with $\omega_0$.*

In view of Lemma 1, we restrict our attention to the classes $A_n(\mu)$ where $\mu$ is not cofinal with $\omega_0$ (this means that $\mu$, though a limit, is not the limit of a countable sequence of smaller ordinals).

## 3. Invariants and a uniqueness theorem

Suppose $(H, G)$ is an $A_n(\mu)$-pair where the limit ordinal $\mu$ is not cofinal with $\omega_0$. It is well known that $G$ is Hausdorff and complete in its $\mu$-topology (the topology obtained by taking the subgroups $G^\sigma$ $(\sigma < \mu)$ as the neighborhoods of the identity). Denoting by $\overline{H}$ the closure of $H$ in $G$, observe $(G/H)^\mu = \overline{H}/H$.

Throughout we adjoin $\infty$ to the ordinals with the convention $\sigma < \infty$ for every ordinal $\sigma$. For any $p$-group $A$, we denote by $f_A$ the extended Ulm-Kaplansky function of $A$. That is, for every ordinal $\sigma \leq \infty$,

$$f_A(\sigma) = \begin{cases} \dim(A^\sigma[p]/A^{\sigma+1}[p]), & \text{if } \sigma < \infty, \\ \dim(A^\infty[p]), & \text{if } \sigma = \infty. \end{cases}$$

Here $A^\infty$ is simply the divisible part of $A$ and it is understood that dimensions are computed over the field with $p$ elements.

With the above notation, for each positive integer $n$ we define a cardinal-valued function $f_n^H$ of $n+1$ ordinal variables as follows:

$$f_1^H(\alpha_1, \alpha_2) = \begin{cases} f_H(\alpha_2), & \text{if } \alpha_1 = 0, \\ f_{\overline{H}/H}(\alpha_2), & \text{if } \alpha_1 = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

For $n \geq 2$, we define

$$f_n^H(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) = \begin{cases} f_H(\alpha_{n+1}), & \text{if } \alpha_1 = \cdots = \alpha_n = 0, \\ f_{n-1}^{\overline{H}/H}(\alpha_2, \ldots, \alpha_{n+1}), & \text{if } \alpha_1 = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

We refer to the function $f_n^H$ as the $A_n(\mu)$-*invariant* of $H$. Since $\overline{H}$ may be viewed as the completion of $H$ in its $\mu$-topology, it is clear that each $f_n^H$ is an isomorphism invariant of $H$, independent of the choice of a containing totally projective group $G$.

**Theorem 2.** *Suppose $\mu$ is a limit ordinal not cofinal with $\omega_0$ and $n$ is a positive integer. Two $A_n(\mu)$-groups $H$ and $K$ are isomorphic if and only if they have the same $A_n(\mu)$-invariants.*

*Proof*. In view of our observations above, we need only show that $H \cong K$ provided that $f_n^H = f_n^K$. Since the case $n = 1$ is the known result for $\mu$-elementary $A$-groups [4, Theorem 3], we may assume $n \geq 2$ and proceed by induction on $n$.

Select totally projective groups $G$ and $T$, each of length $\mu$, such that $(H, G)$ and $(K, T)$ are $A_n(\mu)$-pairs. Since $(G/H)^\mu$ and $(T/K)^\mu$ are $A_{n-1}(\mu)$-groups with the same $A_{n-1}(\mu)$-invariants, the induction hypothesis implies $(G/H)^\mu \cong (T/K)^\mu$. Set

$$A = \oplus_{\aleph_0}\{G \oplus T \oplus (G/H)/(G/H)^\mu \oplus (T/K)/(T/K)^\mu\},$$

and let $L = G \oplus T \oplus A$, a totally projective group of length $\mu$. Identifying $G$ (respectively, $T$) with the first (respectively, second) direct factor of $L$, note that

$$(L/H)/(L/H)^{\mu} \cong (L/K)/(L/K)^{\mu}$$

and that both of these groups are totally projective. In view of the isomorphism $(G/H)^{\mu} \cong (T/K)^{\mu}$, we have $(L/H)^{\mu} \cong (L/K)^{\mu}$. Thus, this latter isomorphism extends to an isomorphism $L/H \cong L/K$, by virtue of Zippin's Theorem for totally projective groups. Note $f_n^H = f_n^K$ includes equality of the Ulm-Kaplansky invariants $f_H$ and $f_K$. Therefore, since $H$ and $K$ are almost balanced subgroups of $L$, the main theorem in [5] implies $H \cong K$. □

To see that Theorem 2 actually gives us something new, the following example shows that there exist $A_2(\mu)$-groups which are not $\mu$-elementary $A$-groups. The reference for the theory of $c$-valuations used below is [5].

**Example.** Let $\mu$ be a limit ordinal not cofinal with $\omega_0$, and let $E$ be a $\mu$-elementary $A$-group which is not totally projective. Select a $p$-group $A$ such that $A^{\mu} \cong E$ and $A/A^{\mu}$ is totally projective. Such an $A$ may be seen to exist, for example, by adapting the portion of the proof of Theorem 4 in [4, p. 516]. Define a $c$-valuation on $A$ by

$$\|a\| = \begin{cases} \alpha + 1, & \text{if } \alpha < \mu \text{ and } a \in A^{\alpha} \backslash A^{\alpha+1}, \\ \mu, & \text{if } a \in A^{\mu} \text{ and } a \neq 0, \\ \infty, & \text{if } a = 0, \end{cases}$$

for every $a \in A$. By Theorem 2.8 in [5], there exists a totally projective group $G$ and an isotype subgroup $H$ of $G$ such that $G/H \cong A$ as $c$-valuated groups, where the $c$-valuation on $G/H$ is the coset valuation. It now easily follows that $(H, G)$ is an $A_2(\mu)$-pair. Moreover, $H$ is not an $A_1(\mu)$-group, since otherwise $\overline{H}/H \cong E$ would be totally projective.

## 4. An application to modular group algebras

Suppose $F$ is a field of characteristic $p > 0$ and $H$ and $K$ are $p$-groups such that the group algebras $FH$ and $FK$ are $F$-isomorphic. As an application of Theorem 2, we prove that if $H$ (but not a priori $K$) is an $A_n(\mu)$-group, for some positive integer $n$ and limit ordinal $\mu$, then $H \cong K$.

Suppose $A$ is a $p$-group. Letting $\mathrm{aug}\colon FA \to F$ denote the augmentation map, we set $I(FA) = \{x \in FA\colon \mathrm{aug}(x) = 0\}$ and $V(FA) = \{x \in FA\colon \mathrm{aug}(x) = 1\}$. Note that $V(FA)$ is a $p$-group and a subgroup of the group of units of $FA$. Moreover, if $H$ is a subgroup of $V(FA)$ and $V(FA)/(1 + FA \cdot I(FH)) \cong V(F(A/H))$. As a direct corollary of Lemmas 1 and 2 in the author's paper [9], we obtain the following result.

**Lemma 3.** *Suppose $F$ is a perfect field of characteristic $p > 0$. If $H$ is a subgroup of a $p$-group $A$ of limit length $\mu$, then $V(FA)$ has length $\mu$ and $H$*

*is almost balanced in* $A$ *if and only if* $1 + FA \cdot I(FH)$ *is almost balanced in* $V(FA)$.

We are now in position to prove our isomorphism result. We state a slightly more general version than is indicated above. As we shall see, this causes no difficulties in the proof.

**Theorem 4.** *Suppose* $R$ *is a commutative ring with 1 and* $p$ *is a prime number such that* $p \cdot 1$ *is not a unit in* $R$. *Suppose further that* $H$ *is a p-group and* $K$ *is an abelian group with* $RH \cong RK$ *as R-algebras. Then, if* $H$ *is an* $A_n(\mu)$*-group for some positive integer* $n$ *and limit ordinal* $\mu$, $H \cong K$.

*Proof.* Let $M$ be a maximal ideal of $R$ containing $p$, and let $F$ be an algebraic closure of $R/M$. Consequently, $F$ is a perfect field of characteristic $p$ and $FH \cong FK$. Since there is an augmentation-preserving isomorphism $FK \to FH$, $V(FH)$ contains an $F$-basis for $FH$ isomorphic to $K$. Therefore, we may assume $K \leq V(FH)$, $FH = FK$ and $I(FH) = I(FK)$.

We now proceed by induction on $n$. Since the case $n = 1$ is the theorem in [10], we may assume $n \geq 2$. Moreover, since $H \cong K$ whenever $H$ is totally projective by [8, Corollary to Theorem 3], we may assume $\mu$ is not cofinal with $\omega_0$ by Lemma 1.

Let $G$ be a totally projective group of length $\mu$ such that $(H, G)$ is an $A_n(\mu)$-pair. By Theorems 1 and 2 in [8], $A = V(FG)$ is totally projective of length $\mu$ and contains $G$ as a direct factor. Thus, $(H, A)$ is an $A_n(\mu)$-pair and $1 + FA \cdot I(FH) = 1 + FA \cdot I(FK)$ is almost balanced in $V(FA)$ by Lemma 3. Thus, a further application of Lemma 3 shows that $K$ is almost balanced in $A$. Note that $F(A/H) \cong FA/FA \cdot I(FH) = FA/FA \cdot I(FK) \cong F(A/K)$. Thus, $F$ perfect implies $F((A/H)^\mu) \cong F((A/K)^\mu)$ and, since $A/(H)^\mu$ is an $A_{n-1}(\mu)$-group, the induction hypothesis implies $(A/H)^\mu \cong (A/K)^\mu$. Moreover, $f_H = f_K$ by [6, Proposition 5]. Therefore, if $K$ were an $A_n(\mu)$-group, $H$ and $K$ would have the same $A_n(\mu)$-invariants. Thus, in view of Theorem 2, the proof will be complete once we have shown that $K$ is an $A_n(\mu)$-group.

As observed above, $K$ is almost balanced in $A$ and $(A/K)^\mu \cong (A/H)^\mu$ is an $A_{n-1}(\mu)$-group. Moreover, the isomorphism $F(A/H) \to F(A/K)$ may be assumed augmentation-preserving. Any such isomorphism restricts to an isomorphism of $F((A/H)^\mu)$ onto $F((A/K)^\mu)$ and maps $I(F(A/H)^\mu)$ onto $I(F(A/K)^\mu)$. Therefore, $F((A/H)/(A/H)^\mu) \cong F(A/H)/(F(A/H) \cdot I(F(A/H)^\mu))$ $\cong F(A/K)/(F(A/K) \cdot I(F(A/K)^\mu)) \cong F((A/K)/(A/K)^\mu)$. Since $(A/H)/(A/H)^\mu$ is totally projective, a further application of the corollary to Theorem 3 in [8] implies $(A/K)/(A/K)^\mu$ is totally projective. Therefore, $(K, A)$ is an $A_n(\mu)$-pair, and the proof is complete. □

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