A CLOSURE THEOREM FOR $\sigma$-COMPACT SUBGROUPS OF LOCALLY COMPACT TOPOLOGICAL GROUPS

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Abstract. We describe the closure of certain subgroups of a locally compact group.

D. Z. Djoković proved the following theorem.

Theorem ([1]). Let $G$ be a real Lie group, $A$ a closed subgroup of $G$, and $B$ an analytic subgroup of $G$. We assume that $B$ normalizes $A$ and that $AB$ is closed in $G$. Then we have

$$B^\sim = (A \cap B)^\sim \cdot B.$$  

In particular, $B$ is closed in $G$ if and only if $A \cap B$ is closed in $G$.

For many interesting applications of the above theorem, we refer to [1] and [2]. In this note, we generalize it into the following theorem.

Theorem. Let $G$ be a locally compact (Hausdorff) topological group. Let $B$ be a $\sigma$-compact subgroup of $G$. Suppose there exists a closed subgroup $A$ of $G$ such that $B$ normalizes $A$ and $BA$ is closed. Then the closure $B^\sim$ of $B$ is the group $(A \cap B)^\sim \cdot B$.

Since an analytic subgroup is $\sigma$-compact, Djoković's result is an immediate consequence of the above theorem.

Our proof is simple, using a known categorical argument for topological groups which we state as a lemma (cf. Theorem 5.29 of [4] for a similar result).

Lemma. Let $F$ be a $\sigma$-compact topological group. If there exists a continuous isomorphism $f$ from $F$ onto a locally compact topological group $H$, then $F$ is locally compact and $f$ is a topological isomorphism, i.e. $f$ is an open map.

Proof. First, we show that $F$ is locally compact. Since $F$ is $\sigma$-compact, there exists a sequence of compact subsets $\{D_i : i = 1, 2, \ldots\}$ of $F$ such that $F = \bigcup_{i=1}^{\infty} D_i$. Since $H = f(F) = \bigcup_{i=1}^{\infty} f(D_i)$, $f(D_i)$ has nonvoid interior $f(D_i)$.

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for some $D_i$ by the Baire category theorem. Then $f^{-1}(f(D_i))^{0}$ is an open subset of $F$ with compact closure. Therefore $F$ is a locally compact group.

Now, we show that $f$ is an open map. Let $U$ be any compact neighborhood of identity $1_F$ of $F$. Let $V$ be a compact neighborhood of $1_F$ of $F$ such that $V = V^{-1} \subset V^2 \subset U$. Since $F$ is $\sigma$-compact, $F = \bigcup_{i=1}^{\infty} x_i V$ where $\{x_i: i = 1, 2, \ldots\}$ is a sequence of elements in $F$. Again, by the Baire category theorem, $f(x_i V)$ has nonvoid interior for some $f(x_i V)$. Since $f(V) = f(x_i)^{-1} \cdot f(x_i V)$, $f(V)$ has a nonvoid interior. Now let $h$ be an interior point of $f(V)$. Let $x = f^{-1}(h)$. Then $f(1_F) = f(x^{-1})f(x) = h^{-1}h \in h^{-1}f(V)^0 \subset f(U)$. Hence $f(1_F) = 1_H \in f(U)^0$, and $f$ is an open map at the identity. Both $F$ and $H$ are homogeneous spaces. It follows that $f$ is an open map. The proof of the lemma is now complete.

**Proof of the theorem.** Without loss of generality, we assume that $G = BA$. $A$ is a closed normal subgroup of $G$. Since $B$ normalizes $A \cap B$, $B$ normalizes $(A \cap B)^-$. Therefore $B \cdot (A \cap B)^-$ is a subgroup of $G$. It is straightforward to check that $[B \cdot (A \cap B)^-] \cap A = (A \cap B)^-$. Let $\phi$ be the inclusion map from $B \cdot (A \cap B)^-$ into $G$. Then we have the continuous isomorphism $\phi'$ induced by $\phi$ from $B \cdot (A \cap B)^-/(A \cap B)^-$ onto $G/A$. Since $B \cdot (B \cap A)^-/(B \cap A)^-$ is the homomorphic image of $B$, it is $\sigma$-compact. Since $G/A$ is locally compact, $\phi'$ is an open map and $B \cdot (A \cap B)^-/(A \cap B)^-$ is locally compact by the above lemma. Since $(A \cap B)^-$ is locally compact, therefore $B \cdot (A \cap B)^-$ is locally compact (cf. [3], Theorem 5.25 of [4], or Theorem 2.2 of [5]). Hence $B \cdot (A \cap B)^-$ is closed. Since $B \subset B \cdot (A \cap B)^- \subset B^-$, so $B^- = B \cdot (A \cap B)^-$. The proof is now complete.

**References**

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