A UNICELLULAR UNIVERSAL QUASINILPOTENT WEIGHTED SHIFT

DOMINGO A. HERRERO

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Abstract. For a suitably chosen sequence of weights \( \{a_n\} \), the unilateral weighted shift \( Q \) on \( l^p \) \( (1 \leq p < \infty) \), defined by \( Qe_n = a_ne_{n+1} \) \( (n \geq 1) \), is a unicellular quasinilpotent operator such that \( Q^k \) is not compact for any power \( k \geq 1 \). As a corollary, several applications to approximation of Hilbert space operators are given.

Inductively, define \( y_1 = 1, y_2 = \frac{1}{4}, y_n = (y_1y_2\cdots y_{n-1})^n \), and let \( \{a_n\} \) be the sequence

\[ y_1, y_2, \ldots, y_9, y_1, y_2, \ldots, y_9, y_1, y_2, \ldots, y_9, \ldots, y_9, 000, y_1, \ldots. \]

Let \( Q \) be the unilateral weighted shift defined by \( Qe_n = a_n e_{n+1} \) \( (n \geq 1) \) with respect to the canonical basis \( \{e_n\}_{n \geq 1} \) of \( l^p \) \( (1 \leq p < \infty) \), and let \( Q_r \) be similarly defined, with the first \( 10^r \) weights replaced by zeros. If \( 10^{r-1} \leq k < 10^r \), then a straightforward calculation shows that

\[ \|(Q_r)^k\| = \text{essential norm}(Q_r)^k = \prod_{j=1}^{k} y_j > 0 \]

(so that \( (Q_r)^k \) is not a compact operator), and therefore the spectral radius of \( Q_r \) cannot exceed

\[ \left( \prod_{j=1}^{k} y_j \right)^{1/k} < (y_k)^{1/k} \to 0 \quad (k \to \infty). \]

Since \( Q - Q_r \) has finite rank and the spectrum of a unilateral weighted shift is always connected, we conclude that

\( Q \) is quasinilpotent and \( Q^k \) is not compact for any \( k \geq 1 \).

(The reader is referred to the monograph [13] for general information about unilateral weighted shifts.)

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It only remains to show that $Q$ is unicellular, that is, the invariant subspace lattice of $Q$ is the chain
\[
\{0\}, \mathcal{M}_t = \bigvee \{e_n\}_{n \geq t} \quad (t \geq 1; \mathcal{M}_1 = l^p).
\]
If $1 \leq m \leq \lfloor n/2 \rfloor$ and $n - m + 1 \leq 10^r < n$, then $n - m + s = 10^r$ for some $s$ ($1 \leq s \leq m - 1$) and
\[
\prod_{j=1}^{m} \frac{\alpha_{n-j+1}}{\alpha_j} = \left( \frac{\alpha_{n-m-s+1} \alpha_{n-m+s+2} \cdots \alpha_n}{\alpha_1 \alpha_2 \cdots \alpha_{m-s}} \right) \left( \frac{\alpha_{n-m+1} \alpha_{n-m+2} \cdots \alpha_{n-m+s}}{\alpha_{m-s+1} \alpha_{m-s+2} \cdots \alpha_m} \right) \leq \frac{\alpha_{n-m+s+1} \alpha_{n-m+2} \cdots \alpha_{n-m+s}}{\alpha_{m-s+1} \alpha_{m-s+2} \cdots \alpha_m} \leq (\gamma (10^r - 10^{r-1}))^{1/(10^r - 10^{r-1})} \leq 4^{-m}
\]
because $m < n - m + s$.

If $1 \leq m \leq \lfloor n/2 \rfloor$ and $10^r < n - m + 1 < n \leq 10^r + 1$, then
\[
\prod_{j=1}^{m} \frac{\alpha_{n-j+1}}{\alpha_j} \leq \begin{cases} 
1, & \text{if } m \leq 10, \\
4^{10-m}, & \text{if } m > 10.
\end{cases}
\]
Thus, we have
\[
\sup_n \sum_{m=0}^{n} \prod_{j=1}^{m} \frac{\alpha_{n-j+1}}{\alpha_j} \leq 2 + 2 \sup_n \sum_{m=1}^{\lfloor n/2 \rfloor} \prod_{j=1}^{m} \frac{\alpha_{n-j+1}}{\alpha_j} \leq 2 + 2 \cdot 4^{10} \sum_{m=1}^{\infty} 4^{-m} = 4^{10} < \infty.
\]

It follows from [4, 11] (or [12]) that $Q$ is a strictly cyclic operator; that is, $l^p = \mathcal{A}(Q)e_1$, where $\mathcal{A}(Q)$ is the uniform closure of the polynomials in $Q$ and $f$. Thus, $\mathcal{A}(Q)$ is a maximal abelian Banach algebra of operators (acting on $l^p$), whose Gelfand spectrum coincides with $\sigma(Q) = \{0\}$ [7, Theorem 4]. It readily follows that $\mathcal{A}(Q)$ has exactly one maximal ideal $\mathcal{M}(Q)$. Under the isomorphism of Banach spaces defined by $A \rightarrow A e_1$ (from $\mathcal{A}(Q)$ onto $l^p$), (closed) ideals of $\mathcal{A}(Q)$ correspond to invariant subspaces of $Q$; in particular, $\mathcal{A}(Q)e_1 = \mathcal{M}$ includes all the nontrivial invariant subspaces of $Q$.

Observe that $Q|\mathcal{M}_t$ is a quasinilpotent unilateral weighted shift with weight sequence $\{\alpha_n = \alpha_{n+1}\}_{n \geq 1}$. A cumbersome calculation (similar to the above one) shows that
\[
\sup_n \sum_{m=0}^{n} \prod_{j=1}^{m} \frac{\alpha_{n-j+1}}{\alpha_j} \leq C_t < \infty,
\]
hence we conclude that every nontrivial invariant subspace of $Q|\mathcal{M}_t$ is a subspace of $\mathcal{M}_{t+1}$ ($t \geq 1$).

It readily follows that
\[
\text{Lat } Q = \{0\}; \mathcal{M}_t, \ t \geq 1.
\]

The existence of these kinds of operators affirmatively answers a question of J. B. Conway (personal communication).
1. The above argument (if $Q|_{M_t}$ is strictly cyclic for all $t \geq 1$, then $Q$ is unicellular) is due to R. Gellar [5].

2. For $1 < p < \infty$, the adjoint of $Q$ is the backward weighted shift $R$, defined by $R e_1 = 0$, $R e_n = \alpha_{n-1} e_{n-1}$ ($n > 1$; $R$ acts on $l^q$, where $\frac{1}{p} + \frac{1}{q} = 1$). Clearly,

$$\text{Lat } R = \left\{ \{0\} ; \bigvee \{ e_n \}_n, t \geq 1 ; l^q \right\} .$$

3. In the Hilbert space case ($p = 2$), $Q$ and $R$ are “universal quasinilpotents”; that is, the closure of the similarity orbit of $Q$ (or $R$) coincides with the set of limits of nilpotent operators; more precisely,

$$\mathcal{S}(Q)^- = \mathcal{S}(R)^- = \{ A \in \mathcal{L}(l^2) : \text{the spectrum } \sigma(A) \text{ and the essential spectrum } \sigma_e(A), \text{ of } A, \text{ are connected sets containing the origin, and } \text{ind}(\lambda - A) = 0 \text{ for each complex } \lambda \text{ such that } \lambda - A \text{ is a semi-Fredholm operator} \}$$

(see [1, 2, 5, 6] or [7, Chapter 8]). (Here $\mathcal{S}(T) := \{ W T W^{-1} : W \text{ is invertible} \}$ denotes the similarity orbit of the operator $T$, and the upper bar indicates norm-closure.)

By using this last observation, several known results in the literature can be given very simple proofs. Two examples:

**Corollary 1** (D. W. Hadwin [6]). The norm-closure of the set of all unicellular operators on a Hilbert space $H$ coincides with

$$\{ A \in \mathcal{L}(H) : \sigma(A) \text{ and } \sigma_e(A) \text{ are connected sets and } \text{ind}(\lambda - A) = 0 \text{ for all } \lambda \in \mathbb{C} \text{ such that } \lambda - A \text{ is semi-Fredholm} \} .$$

**Proof.** Let $W$ denote the above described set. The continuity properties of the Riesz–Dunford functional calculus and the stability properties of the semi-Fredholm operators indicate that every unicellular operator on $H$ is a member of $W$ (see, e.g., [10, Chapter 1]; this inclusion is valid in every Banach space). Since $W$ is a closed subset of $\mathcal{L}(H)$, it must include the closure of the unicellular operators.

On the other hand, if $A \in W$ and $\mu \in \sigma_e(A)$, then $A - \mu \in \mathcal{S}(Q)^-$. Thus,

$$W = \bigcup_{\mu \in \mathbb{C}} \{ \mu + T : T \in \mathcal{S}(Q)^- \} .$$

Since $Q$ is unicellular, we are done. □

A subspace $M$ is stably invariant for $T \in \mathcal{L}(H)$ if, for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\| T - T' \| < \delta$, then $T'$ has an invariant subspace $M'$ satisfying

$$\| P_{M'} - P_{M} \| < \varepsilon ,$$

where $P_M$ denotes the orthogonal projection of $H$ onto the subspace $M$.

**Corollary 2** (C. Apostol, C. Foiaș, and N. Salinas [3]; also see [2, Appendix II]). Assume that $\sigma(T)$ and $\sigma_e(T)$ are connected sets and $\text{ind}(\lambda - T) = 0$ for
each \( \lambda \in \mathbb{C} \) such that \( \lambda - T \) is a semi-Fredholm operator. Then \( T \) is "stably transitive"; that is, the only stably invariant subspaces of \( T \) are the trivial ones \((\{0\} \text{ and } \mathcal{H})\).

**Proof.** Let \( \mu \in \sigma_c(T) \); then

\[
T - \mu \in \mathcal{S}(Q) = \mathcal{S}(R) = 0.
\]

Thus, for each \( \delta > 0 \) there exist \( Q' \) similar to \( Q \) and \( R' \) similar to \( R \) such that

\[
\|T - Q'\| < \delta \quad \text{and} \quad \|T - R'\| < \delta.
\]

Let \( \mathcal{E} \) and \( \mathcal{H} \) be any two nontrivial subspaces of \( \mathcal{H} \) invariant under \( Q' \) and, respectively, under \( R' \). Since \( \mathcal{E} \) is necessarily finite-dimensional and \( \mathcal{H} \) is necessarily of finite codimension, we have \( \|P_{\mathcal{E}} - P_{\mathcal{H}}\| \geq 1 \). Furthermore,

\[
\max\{\|P_{\mathcal{E}} - P_{\mathcal{H}}\|, \|P_{\mathcal{H}} - P_{\mathcal{E}}\|\} \geq 1
\]

for each subspace \( \mathcal{M} \) of \( \mathcal{H} \)!

Therefore, \( T \) cannot have any nontrivial stably invariant subspace. \( \Box \)

It is clear from this last example that the operators \( Q \) and \( R \) can play a very interesting role in the still open problem of characterizing all the stably invariant subspaces of a given operator \( T \).

**References**


Department of Mathematics, Arizona State University, Tempe, Arizona 85287