PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 110, Number 3, November 1990

OSCILLATION IN NONAUTONOMOUS SCALAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

CHENG YUANJI

(Communicated by Kenneth R. Meyer)

ABSTRACT. In this paper, we study the oscillation in the nonautonomous scalar differential equations with deviating arguments and get some oscillation criteria. As an application of the results, we prove a conjecture posed by Brian R. Hunt and James A. Yorke in [1].

1. INTRODUCTION

In the study of oscillation theory for nonautonomous equations with deviating arguments, the situation is much more complicated than that in the autonomous case. The problems can be attached in many different ways. In this article, we attempt to examine the equation

$$x'(t) + \sum_{1}^{n} q_i(t)x(t - T_i(t)) = 0 \tag{1}$$

along the characteristic equation road slightly different from that in [2]. As part of the work in the present paper, we prove a conjecture about (1) posed by Brian R. Hunt and James A. Yorke in [1].

Throughout this paper, we shall assume, as in most papers, that $q_i(t) \geq 0$, $T_i(t) \geq 0$, $\lim_{t\to\infty}(t - T_i(t)) = \infty$, $1 \leq i \leq n$. Furthermore, for a fixed $\delta > 0$ and each t (sufficiently large) there is a $i = i(t) \in \{1, \ldots, n\}$ such that

$$q_i(t) \geq \delta, \qquad T_i(t) \geq \delta. \tag{$*$}$$

Remark. In the present paper, we essentially deal with the equation in which the rate functions depend on past-states of a system. Our results are applicable to equations where the rate functions depend both on past-states as well as on the present state of a system; namely,

$$x'(t) + q(t)x(t) + \sum_{1}^{n} q_i(t)x(t - T_i(t)) = 0, \tag{2}$$

Received by the editors July 7, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34K15.

since the oscillation preserving transformation

$$y(t) = x(t)\exp\left\{\int_{t_0}^{t} q(s)\,ds\right\}$$

will change (2) into a new equation for $y(t)$ of the form (1). Note that $q(t)$ can be any continuous function. (cf. [4, 8].)

2. Main results

In this section we shall state and prove the main results of this paper. The first one is

Theorem 1. *If* $(*)$ *holds and*

$$(**)\qquad \varliminf_{t\to\infty} \inf_{\lambda>0} \lambda^{-1} \sum_{1}^{n} q_i(t)\exp\{\lambda T_i(t)\} > 1\,,$$

then all solutions of (1) *oscillate.*

Proof. Assume the contrary. Then we may have an eventually positive solution $x(t)$ of (1). By virtue of $(*)$, we have, for sufficiently large t, $x'(t)+\delta x(t-\delta) \le 0$. From this inequality we deduce that $x(t)$ is positive and decreasing. Integrating the above inequality over $[t, t+\delta/2]$, we see that $x(t)$ satisfies $x(t) > \delta^2 x(t-\delta/2)/2$.

If $M = 2\delta^{-2}$, then

$$(3)\qquad x(t-\delta/2)/x(t) < M.$$

Let $\lambda(t) = -x'(t)/x(t)$. The function $\lambda(t)$ is positive and continuous, and there are positive constants C and t_0 such that $x(t) = C\exp\{-\int_{t_0}^{t}\lambda(s)\,ds\}$. Furthermore, $\lambda(t)$ satisfies the generalized characteristic equation

$$(4)\qquad \lambda(t) = \sum_{1}^{n} q_i(t)\exp\left\{\int_{t-T_i(t)}^{t} \lambda(s)\,ds\right\}.$$

It follows from (3) that $\int_{t-\delta/2}^{t}\lambda(s)\,ds < \ln M$ which implies that $\varlimsup_{t\to\infty}\lambda(t) < \infty$. On the other hand, it is clear from $(*)$ and (4) that $\lambda(t) \ge \delta$ and

$$0 < \delta \le \lambda_0 = \varliminf_{t\to\infty} \lambda(t) < \infty.$$

Thus, if $\varepsilon \in (0, \delta)$ is fixed and t is sufficiently large, we see that

$$\lambda(t) > \sum_{1}^{n} q_i(t)\exp\{(\lambda_0-\varepsilon)T_i(t)\}.$$

By the definition of λ_0, there is a sequence $\{t_k\}\uparrow\infty$ such that $\lambda(t_k) < \lambda_0+\varepsilon$, whence

$$\begin{aligned}\lambda_0+\varepsilon &> \sum_{1}^{n} q_i(t_k)\exp((\lambda_0-\varepsilon)T_i(t_k)) \\ &> (\lambda_0-\varepsilon)\inf_{\lambda>0}\lambda^{-1}\sum_{1}^{n} q_i(t_k)\exp(\lambda T_i(t_k)).\end{aligned}$$

Letting $k \to \infty$ obtains

$$(\lambda_0 + \varepsilon)/(\lambda_0 - \varepsilon) \geq \varliminf_{t\to\infty} \inf_{\lambda>0} \lambda^{-1} \sum_1^n q_i(t) \exp(\lambda T_i(t)).$$

Since $\varepsilon > 0$ is arbitrary, the above inequality implies that

$$1 \geq \varliminf_{t\to\infty} \inf_{\lambda>0} \lambda^{-1} \sum_1^n q_i(t) \exp(\lambda T_i(t)),$$

which leads to a contradiction of $(**)$ and the proof is complete. □

Remark. In the proof of Theorem 1 we have actually shown that the differential inequality $x'(t) + \sum_1^n q_i(t)x(t - T_i(t)) \leq 0$ has no eventually positive solution if $(*)$ and $(**)$ hold.

As an application of Theorem 1 we shall prove B. R. Hunt and J. A Yorke's conjecture; namely,

Corollary 1. *If there exist constants* q_0, T_0, *for which* $0 \leq q_i(t) \leq q_0$, $0 \leq T_i(t) \leq T_0$, $1 \leq i \leq n$, $t > 0$, *then* $(**)$ *implies that all solutions to* (1) *oscillate.*

Proof. To use Theorem 1, we need only to show that $(**)$ and the fact that $q_i(t)$ and $T_i(t)$ are bounded implies $(*)$ holds.

Without loss of generality, we may assume $q_0 = T_0 = T$. Suppose that $(*)$ doesn't hold; then, for any $m > 0$, all $i \in \{1, \dots, n\}$ and sufficiently large t, we have $T_i(t) < 1/m$, or $q_i(t) < 1/m$. Thus we have the estimate

$$\sum_1^n q_i(t) \exp(\lambda T_i(t)) < n \cdot (Te^{\lambda/m} + m^{-1}e^{\lambda T}).$$

In view of $(**)$, we get $n \cdot \inf_{\lambda>0} \lambda^{-1}(Te^{\lambda/m} + m^{-1}e^{\lambda T}) > 1$.

Set $f_m(\lambda) = \lambda^{-1}(Te^{\lambda/m} + 1/m \cdot e^{\lambda T})$. For each $m > 0$, regarded as a parameter, $f_m(\lambda)$ has a unique extremal point with respect to λ, $\lambda = \lambda_m$, which satisfies

$$T/m(e^{\lambda/m} + e^{\lambda T})\lambda - (Te^{\lambda/m} + m^{-1}e^{\lambda T}) = 0. \tag{5}$$

The minimum of $f_m(\lambda)$ over $(0, \infty)$ is

$$\tilde{f}_m = T/m(e^{\lambda_m/m} + e^{\lambda_m T}). \tag{6}$$

Hence, for all $m > 0$,

$$n \cdot \tilde{f}_m > 1. \tag{7}$$

From (5) we get

$$0 < \lambda_m < 1/T + me^{(1/m-T)\lambda_m}. \tag{8}$$

It follows from (5) that $\lambda_m \to \infty$ and $\lambda_m \approx me^{(1/m-T)\lambda_m}$ as $m \to \infty$. Thus $e^{\lambda_m T} \approx m/\lambda_m \cdot e^{\lambda_m/m}$, $\tilde{f}_m \approx T(1/m + 1/\lambda_m)e^{\lambda_m/m} \to 0$ as $m \to \infty$, which contradicts (7) and we are done. □

Corollary 2. *If* $T_i(t) = T_i$, $1 \le i \le n$, *are constants, and* $(**)$ *is satisfied then all solutions of* (1) *oscillate.*

Proof. It suffices to show that in this special case $(**)$ implies $(*)$. Suppose that this is not true. Then $\underline{\lim}_{t\to\infty} \sum_1^n q_i(t) = 0$. Set $T = \max_i T_i$, then

$$\sum_1^n q_i(t)e^{\lambda T_i} < \sum_1^n q_i(t)e^{\lambda T} \quad \text{and} \quad \inf_{\lambda>0} \lambda^{-1} \sum_1^n q_i(t)e^{\lambda T_i} < e \cdot T \sum_1^n q_i(t).$$

Taking the limit gives a contradiction to $(**)$ and the proof is complete. □

Theorem 2. *If* $\min_{\lambda>0} \max_{[T,t]} \lambda^{-1} \sum_1^n q_i(s)e^{\lambda T_i(s)} \le 1$ *for sufficiently large* T, *then* (1) *is nonoscillatory.*

Proof. Without loss of generality, we assume $T = t_0$. Let $\lambda = \lambda_0(t)$ be the solution of $\lambda = \max_{[t_0,t]} \sum_1^n q_i(s)e^{\lambda T_i(s)}$. It follows from the condition of the theorem that $\lambda_0(t)$ exists. It is easy to verify that $\lambda_0(t) \ge 0$ is nondecreasing. To show that (1) has a nonoscillating solution, it suffices to show that (4) has a continuous solution. To this end, we define the sequence

$$\lambda_{k+1}(t) = \sum_1^n q_i(t) \exp\left\{\int_{t-T_i(t)}^t \lambda_k(s)\,ds\right\}, \qquad k \ge 0,$$

inductively. Then $\lambda_1(t) \le \sum_1^n q_i(t) \exp(\lambda_0(t)T_i(t))$ and

$$\begin{aligned} \lambda_1(t) &\le \max_{[t_0,t]} \lambda_1(s) \le \max_{[t_0,t]} \sum_1^n q_i(s) \exp(\lambda_0(s)T_i(s)) \\ &\le \max_{[t_0,t]} \sum_1^n q_i(s) \exp(\lambda_0(t)T_i(s)) = \lambda_0(t). \end{aligned}$$

Thus, it is easy to show by induction that $\{\lambda_k(t)\}$ is a decreasing sequence whence $0 \le \lambda_k(t) \le \lambda_0(t)$ and $\lambda(t) = \lim_{k\to\infty} \lambda_k(t)$ exists. By the Lebesgue dominate convergence theorem, we infer that $\lambda(t)$ satisfies (4). Consequently, $\lambda(t)$ is continuous and the proof is complete. □

Next we consider the perturbed equation of (1),

$$x'(t) + \sum_1^m \tilde{q}_i(t)x(t - \widetilde{T}_i(t)) = 0. \tag{9}$$

The following result shows that the oscillation, or nonoscillation of (1), is stable under a certain kind of perturbation.

Theorem 3.

(i) *If* $m \ge n$, $\tilde{q}_i(t) \ge q_i^+(t)$ *and* $\widetilde{T}_i(t) \ge T_i(t)$ *for* $1 \le i \le n$, *and* $\tilde{q}_i(t) \ge 0$ *for* $i > n$, *then if* (1) *is oscillatory,* (9) *is also oscillatory;*

(ii) *If* $m \ge n$, $\tilde{q}_i^+(t) \le q_i(t)$ *and* $\widetilde{T}_i(t) \le T_i(t)$ *for* $1 \le i \le n$, *and* $\tilde{q}_i(t) \le 0$ *for* $i > n$, *then if* (1) *is nonoscillatory, so is* (9).

Where $f^+(t) = \max(f(t), 0)$.

Proof. We first prove statement (ii). Let $x_0(t)$ be a positive solution of (1) and $x(t) = x_0(t)y(t)$ be the solution of (9). Then $y(t)$ satisfies

$$\begin{aligned} y'(t) = \frac{1}{x_0(t)} \Bigg[& \sum_1^n q_i(t)x_0(t - T_i(t))(y(t) - y(t - \tilde{T}_i(t)) \\ & + \sum_1^n q_i(t)(x_0(t - T_i(t)) - x_0(t - \tilde{T}_i(t)))y(t - \tilde{T}_i(t)) \\ & + \sum_1^n (q_i(t) - \tilde{q}_i(t))x_0(t - \tilde{T}_i(t))y(t - \tilde{T}_i(t)) \\ & - \sum_{i>n} \tilde{q}_i(t)x_0(t - \tilde{T}_i(t))y(t - \tilde{T}_i(t)) \Bigg] \\ = f(t, y_t) & \end{aligned} \tag{10}$$

We next show that if we choose the initial value (t_0, φ) in such a way that $\varphi(t) > 0$ and is increasing, then $y(t)$ is a positive solution of (10). To this end, for any $T > t_0$, it suffices to show that $y'(t) \geq 0$ on $[t_0, T]$. Consider the perturbed equation of (10),

$$y'(t) = f(t, y_t) + \varepsilon y(t), \qquad \varepsilon > 0. \tag{11$_\varepsilon$}$$

The continuous dependence of the solution on the parameter implies that $y_\varepsilon(t, t_0, \varphi) \to y(t, t_0, \varphi)$ on $[t_0, T]$ uniformly as $\varepsilon \to 0$. Consequently $y'_\varepsilon(t, t_0, \varphi) \to y'(t, t_0, \varphi)$. So it is enough to show that $y_\varepsilon(t, t_0, \varphi) > 0$, $y'_\varepsilon(t, t_0, \varphi) > 0$ on $[t_0, T]$ for each $\varepsilon > 0$. If this is false, put

$$\tilde{t} = \sup\{t : y'_\varepsilon(t, t_0, \varphi) > 0, y_\varepsilon(t, t_0, \varphi) > 0 \text{ on } [t_0, t]\}.$$

We first show that the right-hand set is not empty. Since $q_i(t) \geq 0$, $1 \leq i \leq n$, it follows that $x_0(t)$ is nonincreasing. We note that $\varphi(t)$ is positive and increasing. From $(11)_\varepsilon$, we see that $y'_\varepsilon(t_0) \geq \varepsilon y(t_0) = \varepsilon\varphi(t_0) > 0$ and thus t_0 belongs to the set. From the definition of $\tilde{t}$, it is easy to see that $y'_\varepsilon(\tilde{t}) = 0$, $y'_\varepsilon(t) > 0$ on $[t_0, \tilde{t})$, $y_\varepsilon(t) > 0$ on $[t_0, \tilde{t}]$. On the other hand, we deduce from $(11)_\varepsilon$ that $y'_\varepsilon(\tilde{t}) \geq \varepsilon y_\varepsilon(\tilde{t}) > 0$, and this is a contradiction.

Statement (i) is a consequence of statement (ii), since (1) can be regarded as a perturbation equation of (9). □

We obtain an immediate consequence of Theorem 2:

Corollary 3. *If* $T_i(t) = T_i + T_i^{(0)}(t)$, $q_i(t) = q_i + q_i^{(0)}(t)$, *where* $T_i \geq 0$, $q_i \geq 0$, $1 \leq i \leq n$, *are constants, then*

(i) *If* $T_i^{(0)}(t) \geq 0$, $q_i^{(0)}(t) \geq 0$, *and* $\lambda - \sum_1^n q_i e^{\lambda T_i} = 0$ *has no real roots, then* (1) *is oscillating*;

(ii) *If* $T_i^{(0)}(t) \leq 0$, $q_i^{(0)}(t) \leq 0$, *and* $\lambda - \sum_1^n q_i e^{\lambda T_i} = 0$ *has a real root, then* (1) *is nonoscillating.*

Remark. The above corollary provides a simple way to test the oscillation of (1). In fact, if we set

$$\tau_i = \varliminf_{t\to\infty} T_i(t), \qquad q_i = \varliminf_{t\to\infty} q_i(t)$$
$$T_i = \varlimsup_{t\to\infty} T_i(t), \qquad Q_i = \varlimsup_{t\to\infty} q_i(t)$$

then we see that if $\inf_{\lambda>0} \lambda^{-1} \sum_1^n q_i e^{\lambda\tau_i} > 1$, then (1) is oscillatory; if

$$\inf_{\lambda>0} \lambda^{-1} \sum_1^n Q_i e^{\lambda T_i} < 1,$$

then (1) is nonoscillating.

Consider the advanced differential equation

$$x'(t) = \sum_1^n q_i(t + T_i(t)). \tag{12}$$

We have

Theorem 4. *If the conditions of Theorem* 1 *hold, then all solutions of* (12) *are oscillating.*

The above result can be proved by using the same techniques as those in Theorem 1—with some appropriate modifications. So the proof is omitted.

Furthermore, we may consider the oscillation of differential equations with mixed arguments:

$$x'(t) + \sum_1^m p_j(t)x(t + \sigma_j(t)) + \sum_1^n q_i(t)x(t - T_i(t)) = 0, \tag{13}$$

$$x'(t) = \sum_1^m p_j(t)x(t - \sigma_j(t)) + \sum_1^n q_i(t)x(t + T_i(t)). \tag{14}$$

Corollary 4. *If the conditions in Theorem* 1 *are satisfied and, additionally,* $p_j(t) \ge 0$, $1 \le j \le m$, *then* (13) ((14)) *is oscillating.*

Theorem 5. *If* $(*)$ *holds, and* $p_j(t) \ge 0$, $\sigma_j(t) \ge 0$ *and*

$$\varliminf_{t\to\infty} \inf_{\lambda>0} \lambda^{-1} \left\{ \sum_1^m \tilde{p}_j(t) e^{-\lambda\sigma_j(t)} + \sum_1^n q_i(t) e^{\lambda T_i(t)} \right\} > 1,$$

then (13) ((14)) *is oscillatory, where* $\tilde{p}_j(t) = p_j(t) e^{\delta_0 \sigma_j(t)} M^{-[2\sigma_j(t)/\delta]}$, $M = 2\delta^{-2}$, $\delta_0 < \delta$, *and* $[x] = \inf\{n, \text{ integer } n \ge x\}$.

Proof. If we proceed in exactly the same way as in the proof of Theorem 1, we get

$$\lambda(t) = \sum_1^m p_j(t) \exp\left\{ -\int_t^{t+\sigma_j(t)} \lambda(s)\,ds \right\} + \sum_1^n q_i(t) \exp\left\{ \int_{t-T_i(t)}^t \lambda(s)\,ds \right\} \tag{15}$$

and $\delta \le \lambda_0 = \underline{\lim}_{t\to\infty} \lambda(t) < \infty$. It follows from (3) that

$$\exp\left\{-\int_t^{t+\sigma_j(t)} \lambda(s)\,ds\right\} \ge M^{-[2\sigma_j(t)/\delta]}.$$

Thus, for any $\varepsilon > 0 \ \ (< \delta - \delta_0)$, we deduce from (15) that

$$\begin{aligned}\lambda(t) &\ge \sum_1^m p_j(t) M^{-[2\sigma_j(t)/\delta]} + \sum_1^n q_i(t) e^{(\lambda_0-\varepsilon)T_i(t)} \\ &\ge \sum_1^m \tilde{p}_j(t) e^{-(\lambda_0-\varepsilon)\sigma_j(t)} + \sum_1^n q_i(t) e^{(\lambda_0-\varepsilon)T_i(t)}\end{aligned}$$

The proof is identical to that in the proof of Theorem 1, and is omitted. □

3. Generalizations

When we discussed the oscillation properties of the solutions to (1) in the previous section, we made the assumption $(*)$. For equations with bounded delay and bounded coefficients, we see from Corollary 1 that the assumption $(*)$ is reasonable. For the equations with unbounded delays, or unbounded coefficients, the assumption seems to be strict. The reason is that the good balance between the coefficients and time lags can result in oscillation as well. To relax the assumption $(*)$, we try another approach in this section: time transformation method.

If we can choose a suitable positive function $g(t) \in C(t_0, \infty)$ such that $\int^\infty g(t)\,dt = \infty$ and $\int_{t-T(t)}^t g(s)\,ds$ has some desired properties, then we can make the transformation

$$\begin{cases} s = G(t) = \int_{t_0}^t g(\sigma)\,d\sigma \\ y(s) = x(t) = x(G^{-1}(s)). \end{cases}$$

Under this transformation, we may change the coefficients and delays of (1). We can then study the new equation in y and s instead of studying (1).

To demonstrate this method, we shall study two special kinds of equations with unbounded delays, or unbounded coefficients; namely,

$$x'(t) + \sum_1^n q_i(t) x(\mu_i t) = 0, \tag{16}$$

where $0 < \mu_i < 1$, $1 \le i \le n$, are constants and $q_i(t) \ge 0$, $1 \le i \le n$.

Theorem 6. *If*

$$\underline{\lim}_{t\to\infty} \inf_{\lambda > 0} \lambda^{-1} \sum_1^n t q_i(t) e^{-\lambda \ln \mu_i} > 1 \tag{$\frac{*}{**}$}$$

then all solutions of (16) *oscillate.*

Proof. Choose $g(t) = 1/t$, $t > 1$, then $\int_{\mu_i t}^{t} g(s)\,ds = -\ln \mu_i$, $1 \le i \le n$. Setting $s = \ln t$, $y(s) = x(t) = x(e^s)$. Hence under this transformation (16) is changed into

$$y'(s) + \sum_1^n e^s q_i(e^s) y(s + \ln \mu_i) = 0. \tag{17}$$

Note that $\underline{\lim}_{s\to\infty} \inf_{\lambda>0} \lambda^{-1} \sum_1^n e^s q_i(e^s) e^{-\lambda \ln \mu_i} > 1$ is equivalent to $({}^{*}_{**})$. Thus the assertion follows immediately from Corollary 2. □

Consider an equation of a different type:

$$x'(t) + \sum_1^n q_i(t) x(t - \mu_i/t) = 0, \tag{18}$$

where $\mu_i > 0$, $1 \le i \le n$, are constants and $q_i(t) \ge 0$, $1 \le i \le n$.

Theorem 7. *If*

$$\underline{\lim}_{t\to\infty} \inf_{\lambda>0} \lambda^{-1} \sum_1^n (1/t) q_i(t) e^{\lambda \mu_i} > 1, \tag{${}^{**}_{**}$}$$

then all solutions of (18) *are oscillatory.*

Proof. Choose $g(t) = 2t$, then $\int_{t-\mu/t}^{t} g(s)\,ds = 2\mu - \mu^2/t^2$. And let

$$s = t^2, \qquad y(s) = x(t) = x(s^{1/2}).$$

Therefore, under this transformation, (18) now reads

$$y'(s) + \sum_1^n \frac{1}{2} s^{-1/2} q_i(s^{1/2}) y(s - 2\mu_i + \mu_i^2 s^{-1}) = 0. \tag{19}$$

Observe that

$$\underline{\lim}_{s\to\infty} \inf_{\lambda>0} \lambda^{-1} \sum_1^n \tfrac{1}{2}(s^{-1/2}) q_i(s^{1/2}) e^{2\lambda\mu_i} = \underline{\lim}_{t\to\infty} \inf_{\lambda>0} \lambda^{-1} \sum_1^n (1/t) q_i(t) e^{\lambda\mu_i}.$$

The rest of the proof is almost the same as that in the Theorem 1, and we omit the details. □

Acknowledgment

I would like to thank Matts. Essen for his invaluable suggestions in improving this paper, and I also want to thank the referee for pointing out the small errors in the proof of Theorem 1 in the first version of this paper.

REFERENCES

1. B. R. Hunt and J. A. Yorke, *When all solutions of* $x'(t) = -\sum q_i(t)x(t - T_i(t))$ *oscillate*, J. Differential Equations **53** (1984), 139–145.

2. O. Arino, I. Györi, and A. Jawhari, *Oscillation criteria in delay equations*, J. Differential Equations **53** (1984), 115–123.

3. G. Ladas, Y. G. Sficas, and I. P. Stavroulakis, *Necessary and sufficient conditions for oscillation*, Amer. Math. Monthly **90** (1983), 637–640.

4. G. S. Ladde, *Oscillations caused by retarded perturbations of first order linear ordinary differential equations*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **63** (1977), 351–359.

5. T. Kusano, *On even order functional differential equations with advanced and retarded arguments*, J. Differential Equations **45** (1982), 75–84.

6. G. Ladas and I. P. Stavroulakis, *Oscillation caused by several retarded and advanced arguments*, J. Differential Equations **44** (1982), 134–152.

7. G. S. Ladde, V. Lakshmikantham, and B. G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Marcel Dekker, New York and Basel, 1987.

8. Cheng Yuanji, *Oscillation criteria for first order differential equations with deviating arguments* I, II, J. Math. Anal. Appl. (to appear).

9. M. I. Tramov, *Conditions for oscillatory solutions of first order differential equations with a deviating argument*, Izv. Vyssh. Uchebn. Zaved. Mat. **19** (1975), 92–96.

DEPARTMENT OF MATHEMATICS, UPPSALA UNIVERSITY, THUNBERGSVÄGEN 3, S-752 38 UPPSALA, SWEDEN