JONES AND $Q$ POLYNOMIALS FOR 2-BRIDGE KNOTS AND LINKS

TAIZO KANENOBU

(Communicated by Frederick R. Cohen)

Dedicated to Professor Yoko Tao on her sixtieth birthday

Abstract. It is known that the $Q$ polynomial of a 2-bridge knot or link can be obtained from the Jones polynomial. We construct arbitrarily many 2-bridge knots or links with the same $Q$ polynomial but distinct Jones polynomials.

The Jones polynomial $V_L(t) \in \mathbb{Z}[t^\pm 1/2]$ [5] is an invariant of the isotopy type of an oriented knot or link $L$ in the 3-sphere. The writhe $w(D)$ of an oriented planar diagram $D$ of $L$ is the sum of the signs at all the crossings of $D$, according to the convention explained in Figure 1 (p. 836). Let $|D|$ be a diagram $D$ with its orientation unknown. Then Kauffman's bracket polynomial $\langle D \rangle \in \mathbb{Z}[A^\pm 1]$ [10] of $|D|$, which is a regular isotopy invariant, is defined by

\[ \langle 0 \rangle = 1 \text{ for a simple closed curve } 0, \]
\[ \langle D' \rangle = (-A^2 - A^{-2}) \langle D \rangle, \]
\[ \langle D_\pm \rangle = A^{\pm 1} \langle D_0 \rangle + A^{\mp 1} \langle D_\infty \rangle, \]

where $|D'|$ is the disjoint union of $|D|$ and a simple closed curve and $|D_\pm|$ are identical diagrams except near one point, where they are as in Figure 2 (p. 836). Then the Jones polynomial can be defined using the formula

\[ V_L(A^4) = (-A^3)^{-w(D)} \langle D \rangle. \]

The $Q$ polynomial $Q_L(x) \in \mathbb{Z}[x^\pm 1]$ [1, 4] is an invariant of the isotopy type of an unoriented knot or link $|L|$ in the 3-sphere, which is defined by the following formulas:

\[ Q_U(x) = 1 \text{ for the unknot } U, \]
\[ Q_{L_+}(x) + Q_{L_-}(x) = x(Q_{L_0}(x) + Q_{L_\infty}(x)), \]

where the links $|L_i|$ have diagrams $|D_i|$ which are as in the above.

Received by the editors July 21, 1989.
1980 Mathematics Subject Classification (1985 Revision). Primary 57M25.
Key words and phrases. 2-bridge knot, 2-bridge link, $Q$ polynomial, Jones polynomial.
The author was partially supported by Grant-in-Aid for Encouragement of Young Scientists (No. 01740057), Ministry of Education, Science, and Culture.
The author [7] discovered a relation between the Jones and $Q$ polynomials of a 2-bridge knot or link (see, for example, [2, Chapter 12]).

**Theorem**. If $L$ is a 2-bridge knot or link, then it holds that

$$Q_L(x) = 2x^{-1}V_L(t)V_L(t^{-1}) + 1 - 2x^{-1},$$

where $x = -t - t^{-1}$. Identically,

$$Q_L(x) = 2x^{-1}\langle D \rangle\langle D! \rangle + 1 - 2x^{-1},$$

where $x = -A^4 - A^{-4}$, $D$ is a diagram for $L$, and $D!$ is a mirror image of $D$, so that $\langle D! \rangle(A) = \langle D \rangle(A^{-1})$.

This theorem implies that the $Q$ polynomial of a 2-bridge knot or link can be deduced from the Jones polynomial. Conversely, even if a $Q$ polynomial of some 2-bridge knot or link is given, we cannot necessarily infer its Jones polynomial. In fact, through a computer calculation of polynomial invariants of 2-bridge knots and links [9], except for a reflection such as right- and left-handed trefoils, we found many pairs of 2-bridge knots and links with the same $Q$ polynomial but distinct Jones polynomials; \{10_{14}, 10_{31}\}, \{10_{19}, 10_{36}\}, and \{9_4, 9_{10}\} are such pairs in the table of [15]. Also it is known that there exist arbitrarily many skein-equivalent 2-bridge knots [8] and links [6], which thus have the same 2-variable Jones, Jones, Alexander, and $Q$ polynomials. (See [12] for the definition of skein equivalence and 2-variable Jones polynomial.)
Generalizing these examples, we prove

**Theorem.** For any positive integer $N$, there exist $N$ sets of $2^N$ 2-bridge knots (resp. links) $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_N$ with $\mathcal{K}_i = \{K_{i1}, K_{i2}, \ldots, K_{i2^N}\}$ such that

(i) all the knots (resp. links) in the union $\bigcup_{i=1}^{N} \mathcal{K}_i$ share the same $Q$ and Alexander (resp. 2-variable Alexander) polynomials;

(ii) all the knots (resp. links) in each $\mathcal{K}_i$ are skein-equivalent, and so they have the same 2-variable Jones, Jones, and Alexander (resp. reduced Alexander) polynomials; and

(iii) all the knots (resp. links) $K_{11}, K_{21}, \ldots, K_{N1}$ have mutually distinct Jones polynomials.

1. Preliminaries

Let $\alpha$ be a 3-braid. We denote a 3-braid $\alpha S_1^n \alpha^{-1}$ and $\alpha S_2^m \alpha^{-1} S_1^n \alpha$, $m$, $n \in \mathbb{Z}$, by $\alpha(n)$ and $\alpha(m, n)$, respectively, where $S_1$ and $S_2$ are elementary 3-braids as shown in Figure 3. Let $G_\alpha$ and $H_\alpha$ be unoriented 2-bridge knot or link diagrams as shown in Figure 4. From [8, Proposition 2.4], we have

**Lemma 1.**

(i) $\langle H_{\alpha(n)} \rangle = A^n [d - \{1 - (-A^{-4})^n\} d^{-1} \langle G_\alpha \rangle \langle G_\alpha \rangle]$, 

(ii) $\langle G_{\alpha(m, n)} \rangle = A^{m+n} \langle G_\alpha \rangle [(-A^{-4})^m + (-A^{-4})^n - 1 + \{1 - (-A^{-4})^m\} \times \{1 - (-A^{-4})^n\} d^{-2} \langle G_\alpha \rangle \langle G_\alpha \rangle]$, 

where $d = -A^2 - A^{-2}$ is the bracket polynomial of a trivial 2-component link diagram without any crossing.

Let $Q_\alpha(x)$, $Q_{\alpha(n)}(x)$, and $Q_{\alpha(m, n)}(x)$ be the $Q$ polynomials of unoriented 2-bridge knots or links with diagrams $G_\alpha$, $H_{\alpha(n)}$, and $G_{\alpha(m, n)}$, respectively.
Then we have

**Lemma 2.**

(i) \( Q_{\alpha(n)}(x) = \frac{1}{2}(\sigma_{n+1} - \sigma_{n-1})(\mu - \mu^{-1}Q_{\alpha}(x)^2) + \mu^{-1}Q_{\alpha}(x)^2, \)

(ii) \( Q_{\alpha(1,-1)}(x) = (x^2/4)(Q_{\alpha}(x) + 1)^2(Q_{\alpha}(x) + \mu) - \mu, \)

where \( \sigma_n \in \mathbb{Z}[x^{\pm 1}] \) is the polynomial defined by \( \sigma_{n-1} + \sigma_{n+1} = x\sigma_n, \sigma_0 = 0, \sigma_1 = 1, \) and \( \mu = 2x^{-1} - 1 \) is the \( Q \) polynomial of a trivial 2-component link.

**Proof.**

(i) is Lemma 6.1 of [6].

(ii) By Lemma 1 (ii),

\[
\langle G_{\alpha(1,-1)} \rangle = \langle G_{\alpha} \rangle(-A^{-4} - 1 - A^4 + \langle G_{\alpha} \rangle \langle G_{\alpha}(!) \rangle).
\]

By substituting \( \langle G_{\alpha} \rangle \langle G_{\alpha}(!) \rangle = \frac{3}{2}(Q_{\alpha}(x) + \mu) \) (Theorem *) and \( A^{-4} + A^4 = -x \), this becomes

\[
\langle G_{\alpha(1,-1)} \rangle = \frac{x}{2} \langle G_{\alpha} \rangle \langle Q_{\alpha}(x) + 1 \rangle,
\]

from which we have

\[
\langle G_{\alpha(1,-1)}(!) \rangle = \frac{x}{2} \langle G_{\alpha}(!) \rangle \langle Q_{\alpha}(x) + 1 \rangle.
\]

Substituting these formulas into \( \langle G_{\alpha(1,-1)} \rangle \langle G_{\alpha(1,-1)}(!) \rangle = \frac{x}{2}(Q_{\alpha(1,-1)}(x) + \mu) \) (Theorem *), we obtain (ii).

Using Lemma 2, we can prove the following by induction:

**Lemma 3.**

\[
Q_{\alpha(m,n)}(x) = \frac{1}{4\mu^2}(\sigma_{m+1} - \sigma_{m-1} - 2)(\sigma_{n+1} - \sigma_{n-1} - 2)Q_{\alpha}(x)^3
- \frac{1}{4}x(x+2)\sigma_m\sigma_nQ_{\alpha}(x)^2
+ \left\{1 - \frac{1}{4}(\sigma_{m+1} - \sigma_{m-1} - 2)(\sigma_{n+1} - \sigma_{n-1} - 2)\right\}Q_{\alpha}(x)
+ (\mu^2/4)x(x+2)\sigma_m\sigma_n.
\]
Since $\sigma_{n+1} - \sigma_{n-1} = \sigma_{n+1} - \sigma_{n-1}$ and $\sigma_m \sigma_n = \sigma_m \sigma_n$, we have from this

**Lemma 4.** Let $\alpha$ and $\beta$ be 3-braids. If $Q_\alpha(x) = Q_\beta(x)$, then $Q_{\alpha(m, n)}(x) = Q_{\alpha(-m, -n)}(x) = \frac{1}{x} Q_{\alpha(-m, -n)}(x) = Q_{\beta(-m, -n)}(x)$.

**Remark 1.** We can also prove the following by induction:

\[(Q_\alpha(x) + \mu)(Q_{\alpha(m, n)}(x) + Q_{\alpha(m+n)}(x)) = (Q_{\alpha(m)}(x) + Q_\alpha(x))(Q_{\alpha(n)}(x) + Q_\alpha(x)).\]

**Remark 2.** Let $\alpha = S_2^2 S_1$. Then the 2-bridge knots and links $10_{14}, 10_{31}, 10_{9}, 10_{36}, 9_2, 9_6$ given above have diagrams $G_{\alpha(2,1)}, G_{\alpha(2,-1)}, G_{\alpha(2,1)}$, $G_{\alpha(2,-1)}, G_{\alpha(1,1)}, G_{\alpha(-1,-1)}$, respectively.

Suppose that $\alpha$ is a pure 3-braid and $m, n$ are even integers. Let $\nabla_\alpha(z)$ and $\nabla_{\alpha(m, n)}(z)$ be the Conway polynomials of the 2-bridge knots with diagrams $G_{\alpha}$ and $G_{\alpha(m, n)}$, respectively. (See [3]. Substituting $z = t_1 - t_2^{-1}$, we obtain the Alexander polynomials.) Then we can prove the following by induction:

**Lemma 5.** $\nabla_{\alpha(m, n)}(z) = \frac{m n}{4} \nabla_\alpha(z)^3 + \nabla_\alpha(z)$.

Suppose that $\alpha$ is a pure 3-braid and $m, n$ are even integers. Let $\nabla_\alpha(t_1, t_2)$ and $\nabla_{\alpha(m, n)}(t_1, t_2)$ be the Conway potential function of the 2-bridge links with diagrams $H_{\alpha}$ and $H_{\alpha(m, n)}$, respectively, where the links are oriented so that they coincide if $mn = 0$. (See [3]. Substituting $t_i$ for $t_i^{1/2}, i = 1, 2$, we obtain the 2-variable Alexander polynomial.) Then we can prove the following by induction:

**Lemma 6.**

\[
\nabla_{\alpha(m, n)}(t_1, t_2) = \frac{mn}{4}(t_1 - t_1^{-1})^2(t_2 - t_2^{-1})\nabla_\alpha(t_1, t_2)^3 + \nabla_\alpha(t_1, t_2).
\]

### 2. Proof of theorem

**Knot Case.** Let $\alpha = S_2^2 S_1$ and $e_i u_i$ be either $(2e_i, 4e_i)$ or $(4e_i, 2e_i)$, $e_i = \pm 1$. We define a pure 3-braid $\alpha[e_1 u_1, e_2 u_2, \ldots, e_p u_p]$ by

\[
\alpha[e_1 u_1, e_2 u_2, \ldots, e_i u_i, e_{i+1} u_{i+1}] = (\alpha[e_1 u_1, e_2 u_2, \ldots, e_i u_i])(e_{i+1} u_{i+1}),
\]

where we interpret $\alpha[e_1 u_1, e_2 u_2, \ldots, e_p u_p]$ as $\alpha$ if $p = 0$. Let $K_{\alpha[e_1 u_1, e_2 u_2, \ldots, e_p u_p]}$ be the oriented 2-bridge knot with diagram $G_{\alpha[e_1 u_1, e_2 u_2, \ldots, e_p u_p]}$, so $K_{\alpha}$ is $S_2$ in the table of [15]. Let $H_{\alpha : e_1, e_2, \ldots, e_p}$ be the set of $2^p$ knots $K_{\alpha[e_1 u_1, e_2 u_2, \ldots, e_p u_p]}$, $e_i u_i = (2e_i, 4e_i)$ or $(4e_i, 2e_i)$, and $H_{\alpha, p}$ be the union $\bigcup_{e_i = \pm 1} H_{\alpha : e_1, e_2, \ldots, e_p}$, so $H_{\alpha, 0} = \{K_{\alpha}\}$. Then all the knots in $H_{\alpha : e_1, e_2, \ldots, e_p}$ are skein-equivalent by [8, Propositions 3.1 and 3.2], and mutually nonisotopic by [8, Lemma 3.1]. All the knots in $H_{\alpha, p}$ share the same $Q$ and Alexander polynomials by Lemmas 4 and 5. Since for a 2-bridge knot the minimal crossing number equals the maximal degree of the $Q$ polynomial plus one [11, 13], they have the same
minimal crossing number, which we denote by $c_{\alpha,p}$, so $c_{\alpha,0} = 5$. We denote the Jones polynomial of $K_{a[e_1u_1, e_2u_2, \ldots, e_pu_p]}$ (resp. $K_{a}$) by $V_{\alpha; e_1, e_2, \ldots, e_p}(t)$ (resp. $V_{\alpha}(t) = t - t^2 + 2t^3 - t^4 + t^5 - t^6$) and the writhe of $G_{a[e_1u_1, e_2u_2, \ldots, e_pu_p]}$ (resp. $G_{a}$) by $w_{\alpha; e_1, e_2, \ldots, e_p}$ (resp. $w_{\alpha} = -6$). From Lemma 1 (ii), we have

$$\langle G_{a[e_1u_1, e_2u_2, \ldots, e_pu_p]} \rangle = \langle G_{a[e_1u_1, e_2u_2, \ldots, e_pu_p]} \rangle \{A^{-16\epsilon} + A^{-8\epsilon} - 1 + (A^{-16\epsilon} - 1)(A^{-8\epsilon} - 1)\}d^2$$

Because $V_{\alpha; e_1, e_2, \ldots, e_p}(A^4) = (-A^3)^{-w_{\alpha; e_1, e_2, \ldots, e_p}}(G_{a[e_1u_1, e_2u_2, \ldots, e_pu_p]})$ and $w_{\alpha; e_1, e_2, \ldots, e_p} = w_{\alpha} - 4(e_1 + \cdots + e_p)$, we have

$$V_{\alpha; e_1, e_2, \ldots, e_p}(t) = t^6 V_{\alpha; e_1, e_2, \ldots, e_p}(t) \{t^6 + t^3 - t^5 + (t^6 - 1)(t^2 - 1)\} V_{\alpha; e_1, e_2, \ldots, e_p}(t^{-1}) \}.$$  

Let $R_{\alpha; e_1, e_2, \ldots, e_p}$ and $r_{\alpha; e_1, e_2, \ldots, e_p}$ be the maximal and minimal degrees of $V_{\alpha; e_1, e_2, \ldots, e_p}(t)$. Then $c_{\alpha,p} = R_{\alpha; e_1, e_2, \ldots, e_p} - r_{\alpha; e_1, e_2, \ldots, e_p}$ [10, 14, 16]. It is easy to see that

$$R_{\alpha; e_1, e_2, \ldots, e_p} = R_{\alpha; e_1, e_2, \ldots, e_p} + c_{\alpha,p} + 3\epsilon + 2,$$

$$r_{\alpha; e_1, e_2, \ldots, e_p} = R_{\alpha; e_1, e_2, \ldots, e_p} - 2c_{\alpha,p} + 3\epsilon - 2.$$  

Thus we have $c_{\alpha,p+1} = 3c_{\alpha,p} + 4 = 3^{p+1}(c_{\alpha,0} + 2) - 2 = 7 \cdot 3^{p+1} - 2$, and $R_{\alpha; e_1, e_2, \ldots, e_p} = R_{\alpha} + (c_{\alpha,0} + 2)(3^p - 1)/2 + 3(\epsilon_1 + \cdots + \epsilon_p) = (7 \cdot 3^p + 5)/2 + 3(\epsilon_1 + \cdots + \epsilon_p)$, which shows that at least $p + 1$ polynomials in $V_{\alpha; e_1, e_2, \ldots, e_p}(t)$'s are distinct.

**Remark 3.** If $\alpha, e_1 = \{K_{a(2e_1, 4e_1)}, K_{a(4e_1, 2e_1)}\}$, $c_{\alpha,1} = 19$, $R_{\alpha; e_1} = 3\epsilon_1 + 13$, $r_{\alpha; e_1} = 3\epsilon_1 - 6$.

$$\mathcal{K}_{\alpha; e_1, e_2} \{K_{a(2e_1, 4e_1)(2e_2, 4e_2)}, K_{a(2e_1, 4e_1)(4e_2, 4e_2)}, K_{a(4e_1, 2e_1)(2e_2, 4e_2)}, K_{a(4e_1, 2e_1)(4e_2, 2e_2)}\},$$

$c_{\alpha,2} = 65$, $R_{\alpha; e_1, e_2} = 34 + 3(\epsilon_1 + \epsilon_2)$. We can see by a computer calculation that $V_{\alpha; e_1, e_2}(t) \neq V_{\alpha; e_1, e_2}(t)$.

**Link case.** We change the definition of the pure 3-braid $\alpha[e_1u_1, e_2u_2, \ldots, e_pu_p]$ as follows:

$$\alpha = S_2^2 S_1^{-2} \alpha[e_1u_1, e_2u_2, \ldots, e_pu_p] \beta S_1^{2n} \beta^{-1} S_1^{2n} \beta,$$

where $\beta = [e_1u_1, \ldots, e_pu_p]$ and $e_iu_i = (2m, 2n)$. Consider the oriented 2-bridge link $L_{\alpha[e_1u_1, e_2u_2, \ldots, e_pu_p]}$ having diagram $H_{\alpha[e_1u_1, e_2u_2, \ldots, e_pu_p]}$. Then we can prove this case in the same way as the other, using lemmas similar to Lemmas 1-4 and Lemma 6.
Remark 4. If $e_i \epsilon_i = (2e_i, 2e_i)$ and $\alpha = S_2^a S_1^b \ldots S_2^a S_1^b$, $a_i$, $b_i = \pm 2$, then each set $\mathcal{R}_{\alpha; e_1, \ldots, e_n}$ consists of a single fibered 2-bridge knot. See, for example, [6, Lemma 6.3]. Thus we can prove: There exist arbitrarily many fibered 2-bridge knots (resp. links) with the same $Q$ and Alexander polynomials but mutually distinct Jones polynomials.

References


Department of Mathematics, Kyushu University, Fukuoka 812, Japan