NOTES ON RENEWAL SYSTEMS

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Abstract. A renewal system is a symbolic dynamical system generated by free concatenations of a finite set of words. Renewal systems are sofic, but not every sofic shift is topologically conjugate to a renewal system.

For background on symbolic dynamical systems and sofic shifts see, for example, [M] or [BMT]. Let $A$ be a finite alphabet. A subshift $R$ of $A^Z$ is called a renewal system if there is a finite set $W$ of finite strings (words) over $A$ such that each element of $R$ can be obtained as an infinite bilateral concatenation of elements of $W$. This term is due to Roy Adler, whom I thank for suggesting this area of study. The set $W^*$ of finite concatenations of words of $W$ is extensively studied in automata theory, but the dynamical properties of renewal systems are in general not well understood.

It is easy to see that every renewal system is sofic. A subshift $S$ of $A^Z$ is sofic if and only if its language (the set of words appearing in elements of $S$) is regular [W]; the language of $R$ is the set of subwords of $W^*$, which is regular. Alternatively, let the words of $W$ be $a_1^{(i)} \ldots a_k^{(i)}, i = 1, \ldots, k$ and let $x_j^{(i)}, i = 1, \ldots, k, j = 1, \ldots, l_i$ be distinct symbols. The renewal system $x$ generated by the words $x_1^{(i)} \ldots x_k^{(i)}, i = 1, \ldots, k$ is a shift of finite type of a special sort we call a loop system because of the appearance of the associated directed graph (Figure 1). $R$ is the image of $x$ under the 1-block map $x_j^{(i)} \rightarrow a_j^{(i)}$ and hence is sofic.

It is also easy to see that not every sofic shift, or even every shift of finite type, is a renewal system. For example, the finite type shift $\Sigma$ given by the directed graph in Figure 2 (here we take the vertices rather than the edges as the alphabet of our shift) is not a renewal system. A generating set would have to contain words of the form $a^m, b^n$ to produce the sequences $a^\infty, b^\infty$ (here powers denote concatenation); but the word $b^n a^m$ may not appear in $\Sigma$. However, $\Sigma$ is topologically conjugate to a renewal system: the 1-block map $a \rightarrow a, b \rightarrow b$,

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$c \rightarrow b$ has a 2-block inverse, and the image of $\Sigma$ is the renewal system generated by \{a, bb, bbb\}.

The main result of this paper is that not every sofic shift is topologically conjugate to a renewal system. The example given is not exotic: similar arguments will apply to many sofic shifts. But it is not known if every irreducible shift of finite type is conjugate to a renewal system. (For recent work in this direction see [GLS].)

**Example.** The sofic shift $S$ given by the labeled directed graph in Figure 3 is not conjugate to a renewal system.

**Proof.** Suppose $S$ is conjugate to a renewal system $R$. Let the conjugacy $\varphi$ and its inverse be given by block codes with memory and anticipation $n$, which we will also denote by $\varphi$ and $\varphi^{-1}$. Thus $(\varphi(x))_0 = \varphi(x_{-n} \ldots x_n)$ for $(x_i) \in S$. The image of the fixed point $a^\infty$ in $S$ is a fixed point $\hat{a}^\infty$ in $R$, so $\varphi(a^m) = \hat{a}^{m-2n}$ for $m > 2n$, and $\varphi(a^m b a^m)$ has the form $\hat{a}^{m-2n} b_n \hat{a}^{m-2n}$. Since the point $a^\infty b a^\infty$ in $S$ is the unique preimage of $\hat{a}^\infty b_n \hat{a}^\infty$ (where for definiteness we may take these points to have zero coordinate $b$, $b_0$ respectively), we must have $\varphi^{-1}(a^k b_{-n} \ldots b_n \hat{a}^k) = a^k b a^k$ for $k \geq 0$.

Now, the generating set $W$ of $R$ must contain the word $\hat{a}'$ for some $r \geq 1$. Also, some concatenation of words in $W$ must have the form $\hat{a}^s b_{-n} \ldots b_n \hat{a}^t$, $s, t \geq 0$, or it would be impossible to produce the sequence $\hat{a}^\infty b_{-n} \ldots b_n \hat{a}^\infty$. 

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**Figure 1**

**Figure 2**

**Figure 3**
So, \( R \) must contain a point

\[
x = \hat{a}^\infty b_{-n} \cdots b_n \hat{a}^m b_{-n} \cdots b_n \hat{a}^\infty
\]

with \( m \geq 2n \). But then

\[
\varphi^{-1}(x) = \hat{a}^\infty b a^{m+2n} b a^\infty,
\]

which is not a point in \( S \). \( \square \)

Note added in proof. Renewal systems are called finitely generated systems by A. Restivo [R], who shows it is decidable when a sofic shift is itself a renewal system.

References


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