PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 110, Number 3, November 1990

TOPOLOGICAL TRANSITIVITY OF COMPACT ACTIONS ON C^*-ALGEBRAS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We give some conditions under which minimality is equivalent to the other notions in noncommutative ergodic theory and prove that in a compact $\mathbf{C}^*$-dynamical system topological transitivity is invariant under exterior equivalence.

1. INTRODUCTION

Let (A, G, α) be a $\mathbf{C}^*$-dynamical system. We shall denote by A^α the fixed point $\mathbf{C}^*$-subalgebra of (A, G, α) and by $\mathbf{H}^\alpha(A)$ the set of all nonzero α-invariant hereditary $\mathbf{C}^*$-subalgebras of A. In [5], the authors define the $\mathbf{C}^*$-dynamical system (A, G, α) as

(1) *uniquely ergodic* if there exists a unique α-invariant state of A,
(2) *weakly ergodic* if A^α contains only the scalar element,
(3) *topologically transitive* if for any $B_1, B_2 \in \mathbf{H}^\alpha(A)$, their product B_1B_2 is not zero,
(4) *minimal* if $\mathbf{H}^\alpha(A) = \{A\}$.

2. TOPOLOGICAL TRANSITIVITY OF COMPACT ACTIONS

Let A be a $\mathbf{C}^*$-algebra. For each representation $\{\pi, H\}$ of A, we denote by $M(\pi)$ the enveloping von Neumann algebra $\pi(A)''$.

Lemma 2.1. *Let (A, G, α) be a $\mathbf{C}^*$-dynamical system. If there exists a faithful representation π of A such that the induced action $\tilde{\alpha}$ is weakly ergodic on $M(\pi)$, then (A, G, α) is minimal.*

Proof. If $B \in \mathbf{H}^\alpha(A)$, then $\pi(B)$ is an $\tilde{\alpha}$-invariant hereditary $\mathbf{C}^*$-subalgebra of $\pi(A)$. Then by [6, 3.11.10] there exists an $\tilde{\alpha}$-invariant open projection e in $M(\pi)$ such that $\pi(B) = (eM(\pi)e) \cap \pi(A)$. Since $\tilde{\alpha}$ is weakly ergodic on $M(\pi)$, e is the identity of $M(\pi)$. Since π is faithful, (A, G, α) is minimal.

Received by the editors August 29, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L55, 47A35.
This research was supported in part by the KOSEF, 1988–1989.

Let (A, G, α) be a $\mathbf{C}^*$-dynamical system and G be a compact group with the normalized Haar measure dg. Then the map defined by

$$x \to \int_G \alpha_g(x)\, dg$$

is a conditional expection from A onto A^α, denoted by P_0.

Theorem 2.2. *Let (A, G, α) be a uniquely ergodic $\mathbf{C}^*$-dynamical system with a compact group G. Then (A, G, α) is minimal and A is unital.*

Proof. Since P_0 is faithful, $A^\alpha \neq 0$. Every state of A^α can be extended to an α-invariant state on A by the map P_0. Therefore A^α has only one state ω, and so $A^\alpha = \mathbf{C}e$, where e is the unit of A^α. Put $\tilde{\omega} = \omega P_0$. Then the G.N.S. representation (π, H, ξ, U) defined by the α-invariant state $\tilde{\omega}$ is the faithful covariant representation of (A, G, α) such that $\pi\alpha_g(x) = U_g\pi(x)U_g^*$ and $U_g\xi = \xi$ for all $g \in G$, $x \in A$. Since $\tilde{\omega}(y^*xe - y^*x) = 0$ for all $x, y \in A$, we have $(\pi(xe - x)\xi) = 0$ for all $x \in A$. Since $\tilde{\omega}$ is faithful, ξ is a separating vector for $\pi(A)$. Furthermore, since π is faithful, $xe = x$ for $x \in A$. By a similar computation we have $ex = x$. Hence e is the identity of A. Let B be an α-invariant hereditary C^*-subalgebra. If we consider the C^*-dynamical system $(B, G, \alpha|_B)$, then B^α have to contain the identity of A. Hence we have $B = A$.

Corollary 2.3. *Let (A, G, α) be a $\mathbf{C}^*$-dynamical system with a compact group G. Then the following statements are equivalent:*

(1) *(A, G, α) is uniquely ergodic,*
(2) *(A, G, α) is weakly ergodic,*
(3) *(A, G, α) is topologically transitive,*
(4) *(A, G, α) is minimal.*

Proof. (1) implies (4) by Theorem 2.2.

(4) clearly implies (3).

(3) $\Rightarrow$ (2). Suppose that (A, G, α) is not weakly ergodic. Then there exists a nonscalar positive element x in A^α. We can choose the continuous, real-valued functions $f(\lambda)$, $g(\lambda)$ in $C(\mathrm{spec}(x))$ such that $\mathrm{supp}\, f(\lambda) \cap \mathrm{supp}\, g(\lambda) \neq \varnothing$. Let B_1 and B_2 be the α-invariant hereditary $\mathbf{C}^*$-subalgebras generated by $f(x)Af(x)$ and $g(x)Ag(x)$ respectively. Then $B_1B_2 = 0$.

(2) $\Rightarrow$ (1). Since G is compact, $A^\alpha \neq \{0\}$. Since (A, G, α) is weakly ergodic, we can define a linear functional ω on A by $P_0(x) = \omega(x)1_A$. Then this is the unique α-invariant state on A.

Remark 2.4. Let (A, G, α) be a $\mathbf{C}^*$-dynamical system with a compact group G. If (A, G, α) is uniquely ergodic, then the Arveson's spectrum of α is equal to the Conne's spectrum of α.

Let (A, G, α) be a $\mathbf{C}^*$-dynamical system. If α is faithful, then we may assume that G is a subgroup in the automorphism group $\mathrm{Aut}(A)$ of A. In this case, we have the following corollary.

Corollary 2.5. *Let (A, G, α) be a $\mathbf{C}^*$-dynamical system. Assume that the subgroup G in $\mathrm{Aut}(A)$ has compact closure $\overline{G}$ in $\mathrm{Aut}(A)$. Then the following statements are equivalent*:

(1) (A, G, α) *is uniquely ergodic,*
(2) (A, G, α) *is weakly ergodic,*
(3) (A, G, α) *is topologically transitive,*
(4) (A, G, α) *is minimal.*

3. Topological transitivity in the perturbed $\mathbf{C}^*$-dynamics

In [7], the authors showed that two $\mathbf{C}^*$-dynamical systems (A, G, α) and (A, G, β) are exterior equivalent iff there exists a $\mathbf{C}^*$-dynamical system $(A \otimes M_2, G, \gamma)$ such that

$$\gamma_t \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} \alpha_t(x) & 0 \\ 0 & \beta_t(y) \end{pmatrix} \quad \text{for all } x, y \in A, \ t \in G.$$

In [3], Bratteli and Elliot defined a notion of topological transitivity that is essentially equivalent to that of Peligard, which we have been using so far. They require that, for each pair of nonzero elements $x, y \in A$, there exists a $g \in G$ such that $x\alpha_g(y) \neq 0$.

Lemma 3.1. *Let (A, G, α) be a C^*-dynamical system. Let p and q be equivalent projections in the multiplier algebra $M(A^\alpha)$ of the fixed point algebra A^α. If the C^*-dynamical system $(pAp, G, \alpha|_{pAp})$ is topologically transitive, then the C^*-dynamical system $(qAq, G, \alpha|_{qAq})$ is also topologically transitive.*

Proof. Suppose that there exist two elements x, y $(\neq 0)$ in pAp such that $x\alpha_g(y) = 0$ for all $g \in G$. By Lemma 3.1, there exists α-invariant partial isometry v in $M(A)$ such that $v^*v = p$ and $vv^* = q$. Since x, y are contained in pAp, we have

$$0 = x\alpha_g(y) = v^*vxv^*v\alpha_g(y)v^*v \quad \text{for all } g \in G.$$

Since v is fixed by α_G, for all $g \in G$ we get

$$0 = v(v^*vxv^*v\alpha_g(y)v^*v)v^* = qvxv^*q\alpha_g(qvyv^*q).$$

Put $x'' = qvxv^*q$ and $y'' = qvyv^*q$, then x'' and y'' are two nonzero elements in qAq, and $x''\alpha_g(y'') = 0$ for all $g \in G$. Hence $(qAq, G, \alpha|_{qAq})$ is not topologically transitive.

Let (A, G, α) be a C^*-dynamical system with a compact group. In the following we assume that A is represented nondegenerately, faithfully and covariantly on a Hilbert space H. We denote by A'' the von Neumann algebra generated by A. We denote by $Z(A)$ the center of some operator algebra A.

Theorem 3.2. *Let (A, G, α) be a topologically transitive C^*-dynamical system and let G be a compact group. Let (A, G, β) be exterior equivalent to the C^*-dynamical system (A, G, α). If $Z(A''^{\beta}) = Z(A'')^{\beta}$, then the C^*-dynamical system (A, G, β) is also topologically transitive.*

Proof. Let (A'', G, α) and (A, G, β) be W^*-dynamical systems induced by the C^*-dynamical system (A, G, α) and (A, G, β) due to the covariance respectively. Since (A'', G, α) is ergodic, A'' is finite. As the similar method in Proposition 8.11.5 in [6], we consider the system $(A'' \otimes M_2, G, \gamma)$ obtained from the C^*-dynamical system $(A \otimes M_2, G, \gamma)$ in the beginning in this section. Put $p = 1 \otimes e_{11}$ and $q = 1 \otimes e_{22}$, where $\{e_{ij} | i, j = 1, 2\}$ is the matrix unit of M_2. Since $Z(A''^{\beta}) = Z(A'')^{\beta}$, we have $Z(A'' \otimes M_2)^{\gamma} = Z((A'' \otimes M_2)^{\gamma})$. If we consider the canonical central trace of the finite von Neumann algebra A'', then p and q are equivalent in $(A'' \otimes M_2)^{\gamma}$. The above result comes from Lemma 3.1, if we consider two C^*-dynamical systems $(p(A \otimes M_2)p, G, \gamma|_{p(A \otimes M_2)p})$ and $(q(A \otimes M_2)q, G, \gamma|_{q(A \otimes M_2)q})$.

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