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CRICKETS, ZIPPERS, AND THE BERS UNIVERSAL TEICHMÜLLER SPACE

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Dedicated to Lipman Bers on his seventy-fifth birthday

ABSTRACT. This paper contains a simple characterization of the conformal mappings f of the unit disk U whose Schwarzians lie in the closure of the Bers universal Teichmüller space. A second characterization is given and three examples are studied for the case where $f(U)$ is the complement of a quasiarc.

1. INTRODUCTION

Let E denote the Banach space of functions ϕ analytic in the unit disk U with norm

$$\|\phi\|_U = \sup_{z\in U} |\phi(z)|(1-|z|^2)^2, \tag{1.1}$$

and let S_f denote the Schwarzian derivative $(f''/f')' - \frac{1}{2}(f''/f')^2$ of a locally conformal mapping f of a domain in the extended complex plane $\overline{\mathbb{C}}$.

In the 1960s, Bers showed how all Teichmüller spaces $T(G)$ of Fuchsian groups G or Riemann surfaces U/G could be embedded in a very natural manner into a universal Teichmüller space T, the subset of E consisting of Schwarzian derivatives S_f of conformal mappings f of U which have a quasiconformal extension to $\overline{\mathbb{C}}$. Besides providing new insight into the spaces $T(G)$, this embedding raised the interesting question of how T is related to the subset S of E of Schwarzians S_f of *all* conformal mappings f of U. For example, Bers asked if $S = \overline{T}$ and, later, if S is itself connected. It turns out that S is closed in E, that $\operatorname{int} S$ is equal to T [G2] and hence is connected [EE], that $S\backslash\overline{T}$ is not empty [G3], and that S contains isolated points and thus is not connected [T, As].

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In §2 of this paper we give a simple characterization of the conformal mappings of U whose Schwarzians lie in $\overline{T}$; our approach is a modification of a method used in [AG]. We apply this result in §3 to obtain a second characterization for the case where $f(U) = \overline{\mathbb{C}}\backslash\alpha$ and α is a quasiarc. We consider three examples of this situation in §4 and show, in particular, that S_f is not in $\overline{T}$ for a class of quasiarcs α which includes Thurston's simple zippers [T, §3]. We conclude in §5 with a remark on the relation between T and the corresponding space T_1, where S_f is replaced by f''/f'.

2. Closure of T

Throughout this paper, all quasiconformal mappings are assumed to be sense preserving. Given a simply connected domain D of hyperbolic type in $\overline{\mathbb{C}}$, we let ρ_D denote the density of the hyperbolic metric with curvature -4 in D. Next, if f is conformal in D, we set

$$\|S_f\|_D = \sup_{z\in D} |S_f(z)|\rho_D(z)^{-2}. \tag{2.1}$$

This is the counterpart of (1.1) for the domain D.

2.2. **Theorem.** *Suppose that f is a conformal mapping of U onto D. Then S_f is in $\overline{T}$ if and only if for each $K > 1$ there exists a homeomorphism g of D onto a quasidisk such that for each disk B in D, $g|B$ has a K-quasiconformal extension to $\overline{\mathbb{C}}$.*

Proof. Suppose that S_f is in $\overline{T}$ and that B is a disk in D. For each $K > 1$, we can choose a conformal mapping ϕ of U onto a quasidisk with

$$\|S_g\|_B \le \|S_g\|_D = \|S_f - S_\phi\|_U \le 2\frac{K-1}{K+1}$$

where $g = \phi \circ f^{-1}$. Then by [L, Theorems II.4.1 and II.5.1], $g|B$ has a quasiconformal extension to $\overline{\mathbb{C}}$ with complex dilatation μ_g where

$$\|\mu_g\|_{L^\infty} \le \frac{1}{2}\|S_f\|_D \le \frac{K-1}{K+1},$$

hence establishing the necessity.

For the sufficiency, fix $0 < \varepsilon < 1$ and $c \ge 1$, choose $K > 1$ so that

$$\varepsilon = 96c^2\frac{K^2-1}{K^2+1},$$

and let g be the homeomorphism corresponding to K in the hypotheses. Then g is K-quasiconformal in a neighborhood of each point of $D\backslash\{\infty\}$ and hence in D; by the measurable Riemann mapping theorem, there exists an h, K-quasiconformal in $\overline{\mathbb{C}}$, such that $\psi = h \circ g$ is conformal in D.

Fix z_0 in $D\backslash\{\infty\}$ and let $B = B(z_0, r)$ be the disk with center z_0 and radius r where $cr = \operatorname{dist}(z_0, \partial D)$. If g_B is the extension of $g|B$, then

$\psi_B = h \circ g_B$ is K^2-quasiconformal in $\overline{\mathbb{C}}$, ψ_B agrees with ψ in B, and

$$|S_\psi(z_0)|\rho_B(z_0)^{-2} \leq \|S_\psi\|_B \leq 6\frac{K^2-1}{K^2+1} = \frac{\varepsilon c^{-2}}{16}$$

[L, Theorem II.3.2]. The Koebe distortion theorem implies that

$$\rho_D(z_0)^{-1} \leq 4\,\mathrm{dist}(z_0, \partial D) \leq 4c\rho_B(z_0)^{-1}$$

and hence that

$$|S_\psi(z_0)|\rho_D(z_0)^{-2} \leq 16c^2|S_\psi(z_0)|\rho_B(z_0)^{-2} \leq \varepsilon.$$

Thus

$$\|S_f - S_\phi\|_U = \|S_\psi\|_D \leq \varepsilon$$

where $\phi = \psi \circ f$, $\phi(U) = \psi(D)$ is a quasidisk and S_f is in $\overline{T}$. □

2.3. *Remark.* The above proof shows that S_f is in $\overline{T}$ if for some fixed $c \geq 1$ and each $K > 1$ there exists a homeomorphism g of D onto a quasidisk such that $g|B$ has a K-quasiconformal extension to $\overline{\mathbb{C}}$ for all $B = B(z_0, r)$ in D with $cr \leq \mathrm{dist}(z_0, \partial D)$.

For convenience of notation we let S, T, and $\overline{T}$ also denote the *family of images* $f(U)$ when S_f is in S, T, and $\overline{T}$, respectively. With this convention, we see that if g is a Möbius transformation, then D is in $\overline{T}$ whenever $g(D)$ is in $\overline{T}$. We have the following useful extension of this observation:

2.4. **Corollary.** *A domain D is in $\overline{T}$ if for each $K > 1$ there exists a K-quasiconformal self-mapping g of $\overline{\mathbb{C}}$ with $g(D)$ in $\overline{T}$.*

Proof. By a distortion theorem for quasiconformal mappings, we can choose a constant $c \geq 1$ with the following property: If $B = B(z_0, r)$ is in $D \subset \mathbb{C}$ with $cr \leq \mathrm{dist}(z_0, \partial D)$ and if g is a K-quasiconformal mapping of D onto $D' \subset \mathbb{C}$ with $1 \leq K \leq 2$, then $g(B)$ lies inside a disk B' in D'. See [G1, p. 383] or [V1, Theorem 18.1].

Suppose g is K-quasiconformal in $\overline{\mathbb{C}}$ with $1 < K \leq 2$ and $D' = g(D)$ in $\overline{T}$; as noted above, we may assume that D and D' lie in $\mathbb{C}$. By Theorem 2.2, there exists a homeomorphism h of D' onto a quasidisk D'' such that $h|B'$ has a K-quasiconformal extension to $\overline{\mathbb{C}}$ for each disk B' in D'. Then $h \circ g(D) = D''$, and $h \circ g|B$ has a K^2-quasiconformal extension to $\overline{\mathbb{C}}$ for $B = B(z_0, r)$ in D with $cr \leq \mathrm{dist}(z_0, \partial D)$. □

3. The complement of a quasiarc

Most of the known domains in $S\backslash\overline{T}$ are of the form $D = \overline{\mathbb{C}}\backslash\alpha$, where α is a quasiarc, i.e., the image of the closed positive half $I = [0, \infty]$ of the extended real axis R under a quasiconformal mapping of $\overline{\mathbb{C}}$. (cf. [As, G3, T]. But see also [F].) We use Theorem 2.2 to derive the following criterion for the complement of a quasiarc to be in $\overline{T}$.

3.1. **Cricket Theorem.** *Suppose that α is a subarc of a quasicircle γ in $\overline{\mathbb{C}}$. Then $D = \overline{\mathbb{C}}\backslash\alpha$ is in $\overline{T}$ if and only if for each $K > 1$ there exists a sense preserving K-quasiconformal self-mapping h of $\overline{\mathbb{C}}$ such that $h(z) = z$ in $\gamma\backslash\alpha$ and $h(\alpha) \cup \alpha$ is a quasicircle.*

3.2. *Remark.* If we think of γ as a band of spring steel, Theorem 3.1 says that $D = \overline{\mathbb{C}}\backslash\alpha$ is in $\overline{T}$ if and only if the part corresponding to α can be snapped by small deformations into positions which differ markedly from the original, while the remainder of the band remains fixed. This similarity to a metal toy cricket is the reason for the Theorem's name.

Proof of the necessity in Theorem 3.1. Let G_1 and G_2 denote the components of $\overline{\mathbb{C}}\backslash\gamma$. Then there exists a constant $a = a(\gamma) > 0$ such that each f conformal in G_j with $\|S_f\|_{G_j} < a$ has a quasiconformal extension g_j to $\overline{\mathbb{C}}$ with

$$\|\mu_{g_j}\|_{L^\infty} \leq \tfrac{1}{a}\|S_f\|_{G_j} \tag{3.3}$$

[L, Theorem II.4.1].

Fix $K > 1$, and choose $0 < \varepsilon < 1$ so that $(1+\varepsilon)/(1-\varepsilon) \leq \sqrt{K}$. Because D is in $\overline{T}$, there exists a conformal mapping f of D onto a quasidisk D' with

$$\|S_f\|_{G_j} \leq \|S_f\|_D \leq a\varepsilon .$$

By (3.3), $f|G_j$ has a quasiconformal extension g_j to $\overline{\mathbb{C}}$ with maximal dilatation $K(g_j) \leq \sqrt{K}$. Then $h = g_2^{-1} \circ g_1$ is K-quasiconformal in $\overline{\mathbb{C}}$, h fixes each point of $\gamma\backslash\alpha = \beta$, and

$$g_1(\alpha) \cup g_2(\alpha) = \partial D' .$$

Hence $h(\alpha) \cup \alpha = g_2^{-1}(\partial D')$ is a quasicircle. □

The proof for the sufficiency in Theorem 3.1 requires several preliminary results.

3.4. **Lemma.** *For each $0 < a < \infty$, there exists $0 < b = b(a) < \infty$ with the following property: If g is K-quasiconformal in $\overline{\mathbb{C}}$, with $1 \leq K \leq e^{a/b}$, and if g fixes z_1, z_2, ∞, then*

$$|g(z) - z| \leq b|z_1 - z_2| \log K \tag{3.5}$$

for z in $B(z_1, a|z_1 - z_2|)$.

Proof. The argument is similar to that for Lemma 7 in [G3]. Let ρ_G and h_G denote the hyperbolic density and metric with curvature -4 in $G = \overline{\mathbb{C}}\backslash\{0, 1, \infty\}$ and choose b so that

$$1 = 2b \inf\{\rho_G(z) \colon z \in G \cap \overline{B}(0, 2a)\} .$$

Then, by a theorem of Teichmüller [Ah, pp. 53–61],

$$h_G(g(z), z) \leq \tfrac{1}{2}\log K$$

for each K-quasiconformal mapping g of $\overline{\mathbb{C}}$ which fixes $0, 1, \infty$.

Suppose that g satisfies the above hypotheses with $z_1 = 0$ and $z_2 = 1$. Fix $z \in G \cap B(0, a)$, and let γ denote the hyperbolic geodesic joining z and $g(z)$ in G and $\ell(\gamma)$ the euclidean length of γ. Then $\gamma \subset \overline{B}(0, 2a)$ since otherwise

$$a < \ell(\gamma \cap \overline{B}(0, 2a)) \leq 2b \int_\gamma \rho_G \, ds = 2bh_G(g(z), z) \leq b \log K,$$

contradicting our choice of K. Thus

$$|g(z) - z| \leq \ell(\gamma) \leq 2b \int_\gamma \rho_G \, ds \leq b \log K,$$

and we obtain (3.5) for the case where $z_1 = 0$ and $z_2 = 1$. The general case follows from applying what was proved above to

$$h(w) = \frac{g(z) - z_1}{z_2 - z_1}, \qquad z = w(z_2 - z_1) + z_1. \quad \square$$

A continuous map g of $E \subset \mathbb{C}$ into $\mathbb{C}$ is *s-quasisymmetric* [V2], $0 < s < \infty$, if, for each three points z, z', z'' in E,

$$s \leq \left| \frac{z' - z}{z'' - z} \right| \leq \frac{1}{s} \text{ implies that } \left| \frac{g(z') - g(z)}{g(z'') - g(z)} \right| \leq \left| \frac{z' - z}{z'' - z} \right| + s.$$

3.6. **Lemma.** *For each $s > 0$ there exist $t > 0$ and $K > 1$ with the following property: If g is K-quasiconformal and satisfies $|g(z) - z| \leq tr$ in $B(z_0, 2r)$, then g is s-quasisymmetric in $\overline{B}(z_0, r)$.*

Proof. Since the composition of an s-quasisymmetric map with a euclidean similarity is s-quasisymmetric, we may assume $z_0 = 0$ and $r = 1$.

Suppose that the conclusion of Lemma 3.6 does not hold for some $s > 0$. Then for each integer j there exists a homeomorphism g_j which is K_j-quasiconformal and satisfies $|g_j(z) - z| \leq t_j$ in $B(0, 2)$, where $K_j \to 1$ and $t_j \to 0$. In addition, there exist points z_j, z_j', z_j'' in $\overline{B}(0, 1)$ such that

$$s \leq \frac{|z_j' - z_j|}{|z_j'' - z_j|} \leq \frac{1}{s} \quad \text{and} \quad \frac{|w_j' - w_j|}{|w_j'' - w_j|} - \frac{|z_j' - z_j|}{|z_j'' - z_j|} > s, \tag{3.7}$$

where $w_j = g_j(z_j)$, $w_j' = g_j(z_j')$, $w_j'' = g_j(z_j'')$. By passing to a subsequence we may assume that z_j, $w_j \to z_0$, z_j', $w_j' \to z_0'$, z_j'', $w_j'' \to z_0''$. Condition (3.7) then implies that $z_0 = z_0' = z_0''$.

Let $h_j = \psi_j \circ g_j \circ \phi_j^{-j}$, where ϕ_j and ψ_j are euclidean similarities with

$$\phi_j(z_j) = 0, \quad \phi_j(z_j'') = 1, \quad \psi_j(w_j) = 0, \quad \psi_j(w_j'') = 1.$$

Then h_j is K_j-quasiconformal in

$$B_j = \phi_j(B(0, 2)) \supset B(0, r_j), \qquad r_j = |z_j'' - z_j|^{-1}.$$

Since h_j fixes $0, 1$ and omits ∞ in B_j, by passing to a second subsequence we may assume that $|h_j(z)| \to |z|$ locally uniformly in $\mathbb{C}$. Then $|\phi_j(z_j')|$ is bounded in j by the first part of (3.7) and

$$\frac{|w_j' - w_j|}{|w_j'' - w_j|} - \frac{|z_j' - z_j|}{|z_j'' - z_j|} = |h_j(\phi_j(z_j'))| - |\phi_j(z_j')| \to 0,$$

which contradicts the second part of (3.7). □

3.8. **Lemma.** *For each $K > 1$, there exists $s > 0$ with the following property: If g is an s-quasisymmetric homeomorphism of a closed disk B, then $g|B$ has a homeomorphic extension to $\overline{\mathbb{C}}$ which is K-quasiconformal in $\overline{\mathbb{C}}\backslash B$.*

Proof. This follows from [TV, Theorems 5.23 and 2.6]. □

Proof of sufficiency in Theorem 3.1. By performing a preliminary Möbius transformation, we may assume that α has 0 and ∞ as its endpoints. Next, by hypothesis, there exists for each $K > 1$ a K-quasiconformal self-mapping h of $\overline{\mathbb{C}}$ such that $h(z) = z$ for z in $\beta = \gamma\backslash\alpha$, and $h(\alpha) \cup \alpha$ is a quasicircle. Hence h fixes 0 and ∞,

$$h(\alpha) \cap \gamma = h(\alpha) \cap \alpha = \{0, \infty\}$$

and we can label the components G_1 and G_2 of $\overline{\mathbb{C}}\backslash\gamma$ so that $h(\alpha) \subset G_2 \cup \{0, \infty\}$. Then the fact that h is sense preserving and the identity on $\gamma\backslash\alpha$ implies that $h(G_2) \subset G_2$. Hence

$$g(z) = \begin{cases} z & \text{if } z \in G_1 \cup \beta, \\ h(z) & \text{if } z \in G_2 \end{cases}$$

defines a homeomorphism of $D = G_1 \cup \beta \cup G_2$ onto $D' = G_1 \cup \beta \cup h(G_2)$, and $\partial D' = h(\alpha) \cup \alpha$ is a quasicircle.

To complete the proof, it suffices to show that for each $B = B(z_0, r)$ in D with $3r \le \operatorname{dist}(z_0, \partial D)$, $g|B$ has a K_0-quasiconformal extension g_0 to $\overline{\mathbb{C}}$ where $K_0 = K_0(K) \to 1$ as $K \to 1$. If $B \subset G_1$, we may take $g_0(z) = z$ and $K_0 = 1$; if $B \subset G_2$, we may choose $g_0(z) = h(z)$ and $K_0 = K$.

It remains to consider the case where there exists a point $z_1 \in B \cap \beta$. Since β is unbounded, we can choose $z_2 \in \beta$ with $|z_1 - z_2| = r$. Then h is K-quasiconformal in $\overline{\mathbb{C}}$ and fixes z_1, z_2, ∞. Hence, by Lemma 3.4 applied to h in $B(z_1, 3r)$, there exists an absolute constant $b > 0$ such that

$$|g(z) - z| \le |h(z) - z| \le tr, \qquad t = b \log K$$

if $z \in B(z_0, 2r)$ and $1 \le K \le e^{3/b}$. Since g is K-quasiconformal in $B(z_0, 2r)$, Lemma 3.6 implies that g is s-quasisymmetric in $\overline{B}(z_0, r)$, where $s = s(K) \to 0$ as $K \to 1$. Then by Lemma 3.8, $g|B$ has a K_0-quasiconformal extension to $\overline{\mathbb{C}}$ where $K_0 = K_0(s) \to 1$ as $s \to 0$. □

4. Examples

We apply the Cricket Theorem to give two classes of quasiarcs α for which $D = \overline{\mathbb{C}}\backslash\alpha$ is in $\overline{T}$ and a third for which $D = \overline{\mathbb{C}}\backslash\alpha$ is not in $\overline{T}$.

A quasiarc α is *asymptotically conformal* if, for each $K > 1$, $\alpha = g(I)$, where g is a quasiconformal in $\overline{\mathbb{C}}$ and K-quasiconformal in a neighborhood V of $I = [0, \infty]$. If $\alpha \subset \mathbb{C}$ is asymptotically conformal, then

$$\frac{|z_1 - z_2| + |z_2 - z_3|}{|z_1 - z_3|} \to 1 \quad \text{as } |z_1 - z_3| \to 0,$$

for each ordered triple of points z_1, z_2, z_3 in α.

4.1. **Theorem.** *If α is an asymptotically conformal quasiarc, then $D = \overline{\mathbb{C}} \backslash \alpha$ is in $\overline{T}$.*

Proof. Fix $K > 1$. Then there exists g quasiconformal in $\overline{\mathbb{C}}$ with g K-quasiconformal in a neighborhood V of I and $g(I) = \alpha$. Choose $0 < a < \pi$ and $0 < b < a$ so that

$$W = \{z = re^{i\vartheta} : 0 \le r \le \infty, |\vartheta| \le a\} \subset V, \qquad \frac{a}{a-b} = K$$

and set

$$f(re^{i\vartheta}) = re^{i\phi(\vartheta)}, \qquad f(\infty) = \infty, \tag{4.2}$$

where ϕ is piecewise linear and increasing in $[-\pi, \pi]$ with $\phi(\pm\pi) = \pm\pi$, $\phi(\pm a) = \pm a$ and $\phi(0) = b$. Then f is a K-quasiconformal self-mapping of $\overline{\mathbb{C}}$, $f(W) = W$ and $f(z) = z$ in $\overline{\mathbb{C}} \backslash W$.

Let $h = g \circ f \circ g^{-1}$. Then h is K^3-quasiconformal in $g(V)$ and each point of $\overline{\mathbb{C}} \backslash g(W)$ has a neighborhood in which $h(z) = z$. Thus h is K^3-quasiconformal in $\overline{\mathbb{C}}$, $h(\alpha) \cup \alpha$ is a quasicircle and D is in $\overline{T}$ by Theorem 3.1 with $\gamma = g(R)$ and R the real axis. □

An arc α is a graph if for each line λ parallel to some fixed line, the set $\alpha \cap \lambda$ is either connected or empty.

4.3. **Theorem.** *If α is a quasiarc which is a graph, then $D = \overline{\mathbb{C}} \backslash \alpha$ is in $\overline{T}$.*

Proof. We consider here only the case where α is unbounded. Then, by means of a preliminary similarity mapping, we may assume that $\alpha \cap \lambda$ is connected or empty for each line λ parallel to the y-axis, that α has endpoints 0 and ∞, and that α does not meet the left half-plane.

Suppose that there exists a constant $1 \le m < \infty$ such that

$$|z_1 - z_2| \le m|\operatorname{Re}(z_1) - \operatorname{Re}(z_2)| \tag{4.4}$$

for each z_1, z_2 in $\alpha \cap \mathbb{C}$. Given $K > 1$, choose t so that $(1+t)/(1-t) = K$, and let

$$h(z) = \begin{cases} z + it\operatorname{Re}(z) & \text{if } 0 \le \operatorname{Re}(z) < \infty, \\ z & \text{if } -\infty < \operatorname{Re}(z) \le 0, \end{cases}$$

and $h(\infty) = \infty$. Then h is K-quasiconformal in $\overline{\mathbb{C}}$ and (4.4) implies that $h(\alpha) \cup \alpha$ is a quasicircle. Hence $D = \overline{\mathbb{C}} \backslash \alpha$ is in $\overline{T}$ by Theorem 3.1.

For the general case, fix $K > 1$ and let $\beta = (-\infty, 0)$ denote the negative half of the real axis. Then $\gamma = \alpha \cup \beta$ is a quasicircle. Let f map the upper half-plane

H conformally onto the upper component G of $\overline{\mathbb{C}}\backslash\gamma$ so that its homeomorphic extension to $\overline{H}$ fixes ∞. Because α is a graph, $\{z+it\colon z\in G\}\subset G$ for $t>0$. Hence, if we let ϕ map H conformally onto U so that 0 and ∞ correspond to -1 and 1, respectively, we can apply [HS, pp. 313–314] to $\psi = f\circ\phi^{-1}$ to conclude that $\operatorname{Re}(f'(z))\geq 0$ in H.

Now set

$$g(z)=\int_i^z f'(\zeta)^t\,d\zeta,\qquad h=g\circ f^{-1},$$

for $0<t<1$. Then

$$\|S_h\|_G=\|S_f-S_g\|_H\leq(1-t)\sup_{z\in H}\left(|S_f(z)|+\frac{t}{2}\left|\frac{f''(z)}{f'(z)}\right|^2\right)\rho_H(z)^{-2}\leq 24(1-t),$$

h has a K-quasiconformal extension to $\overline{\mathbb{C}}$ for t near 1, and

$$|g'(z)|\leq m\operatorname{Re}(g'(z)),\qquad m=\sec(\pi t/2),$$

in H. If $-\infty<\xi_1<\xi_2<\infty$ and $0<\eta<\infty$, then

$$\begin{aligned}|z_1-z_2|\leq\int_{\xi_1}^{\xi_2}|g'(\xi+i\eta)|\,d\xi\leq m\int_{\xi_1}^{\xi_2}\operatorname{Re}(g'(\xi+i\eta))\,d\xi\\ =m|\operatorname{Re}(z_1)-\operatorname{Re}(z_2)|,\end{aligned}$$

where $z_j=g(\xi_j+i\eta)$, and letting $\eta\to 0$ yields (4.4) for z_1, z_2 in $h(\alpha)\cap\mathbb{C}$. Thus $h(\overline{\mathbb{C}}\backslash\alpha)$ is in $\overline{T}$ and hence so is $\overline{\mathbb{C}}\backslash\alpha$ by Corollary 2.4. □

Theorem 4.3 allows us to characterize those quasiarcs, which are invariant with respect to a homothety $f(z)=az$, with complements in $\overline{T}$.

4.5. **Corollary.** *Suppose that α is a quasiarc, that 0 is an interior point of α, and that $f(\alpha)\subset\alpha$ for some a in $B(0,1)\backslash\{0\}$. Then $D=\overline{\mathbb{C}}\backslash\alpha$ is in $\overline{T}$ if and only if α is a graph.*

Proof. The necessity follows from [As, Theorem 4.1] and the sufficiency from Theorem 4.3. □

The consideration of quasiarcs that are invariant with respect to a translation $f(z)=z+a$ leads us to the notion of an interlocking sequence. Given any triple of points z_{j-1}, z_j, z_{j+1} in $\mathbb{C}$ we let

$$r(z_j)=\frac{z_{j+1}-z_j}{z_{j-1}-z_j}.$$

Suppose next that α is a subarc of a quasicircle γ which has 0 and ∞ as its endpoints. A sequence of distinct points z_j in α is *interlocking* if there exist constants $0<a,b<\infty$, disjoint neighborhoods V_j of z_j, and a component G of $\overline{\mathbb{C}}\backslash\gamma$ with the following properties:

$$|r(z_j)|\leq a \text{ for } j\geq 2. \tag{4.6}$$

$$(4.7)\quad \begin{cases} \text{If } w_{j-1} \in G \cap V_{j-1},\ w_j \in G \cap V_j,\ w_{j+1} \in G, \text{ and } |r(z_j) - r(w_j)| \le b \\ \text{for some } j \ge 2, \text{ then } w_{j+1} \in V_{j+1}. \end{cases}$$

$$(4.8)\qquad \operatorname{dia}(V_j)/|z_j| \to 0 \quad \text{as } j \to \infty.$$

Here $\operatorname{dia}(V_j)$ denotes the euclidean diameter of V_j.

4.9. **Theorem.** *If α is a quasiarc that contains an interlocking sequence, then $D = \overline{\mathbb{C}}\backslash\alpha$ is not in $\overline{T}$.*

Proof. By hypothesis there exist points z_j in α, neighborhoods V_j of z_j, and a component G of $\overline{\mathbb{C}}\backslash\gamma$ which satisfy (4.6) through (4.8); here γ is any quasicircle which contains α.

Suppose that $D = \overline{\mathbb{C}}\backslash\alpha$ is in $\overline{T}$. Then for each $K > 1$ there exists a K-quasiconformal self-mapping h of $\overline{\mathbb{C}}$ with $h(z) = z$ in $\gamma\backslash\alpha$ and $h(\alpha) \cup \alpha$ a quasicircle. By replacing h by h^{-1} if necessary, we may assume that

$$w_j = h(z_j) \in h(\alpha\backslash\{0, \infty\}) \subset G, \qquad j \ge 1.$$

Next, by Lemma 3.4, we can choose $K > 1$ so that $w_1 \in V_1$ and $w_2 \in V_2$ and so that each K-quasiconformal self-mapping g of $\overline{\mathbb{C}}$ which fixes $0, 1, \infty$ satisfies

$$(4.10)\qquad |g(w) - w| \le b,$$

for w in $B(0, a)$. Then (4.6) and (4.10) applied to

$$g(w) = \frac{h(z) - w_j}{w_{j-1} - w_j}, \qquad z = w(z_{j-1} - z_j) + z_j,$$

yield

$$|r(w_j) - r(z_j)| = |g(r(z_j)) - r(z_j)| \le b, \qquad j \ge 2.$$

Since $w_1 \in V_1$ and $w_2 \in V_2$, (4.7) implies that $w_j \in V_j$ for all j and hence that

$$(4.11)\qquad \lim_{j\to\infty} \frac{|h(z_j) - z_j|}{|z_j|} = 0.$$

Because $h(z_j)$ and z_j separate 0 and ∞ in $h(\alpha) \cup \alpha$, (4.11) then implies that $h(\alpha) \cup \alpha$ is not a quasicircle. □

4.12. **Example.** For $\pi/3 < \vartheta < 2\pi/3$ and $j \ge 1$, let α_j denote the polygonal arc formed by joining successively the points $2j-2$, $2j-1$, $2j-1+e^{i(\pi-\vartheta)}$, $2j+e^{i\vartheta}$, $2j$ with linear segments. Next let

$$\alpha = \left(\bigcup_{j=1}^{\infty} \alpha_j\right) \cup \{\infty\},$$

and let $\beta = (-\infty, 0)$ be the negative half of the real axis. Then $\gamma = \alpha \cup \beta$ is a quasicircle. Moreover, α contains an interlocking sequence when $\vartheta < \pi/2$, α is a graph when $\vartheta \ge \pi/2$, and hence $D = \overline{\mathbb{C}}\backslash\alpha$ is in $\overline{T}$ if and only if $\vartheta \ge \pi/2$.

Proof. Suppose that $\vartheta < \pi/2$. We shall show that the points $z_j = j$ are interlocking in α with

$$a = 1, \quad b = \frac{\cos^2 \vartheta}{8}, \quad c = \frac{2b}{\cos \vartheta}, \quad V_j = B(j, c)$$

and G the upper component of $\overline{\mathbb{C}} \backslash \gamma$. Since (4.6) and (4.8) hold trivially, we need only prove that, if w_{j-1}, w_j, w_{j+1} satisfy the hypotheses in (4.7), then $d_{j+1} < c$ where $d_k = |w_k - k|$.

If j is even, then

$$\begin{aligned} d_{j+1} - 2d_j - d_{j-1} &\le |w_{j+1} - 2w_j + w_{j-1}| \\ &= |r(w_j) - r(z_j)|\,|w_j - w_{j-1}| \\ &\le b|w_j - w_{j-1}| \le b(d_j + 1 + d_{j-1}), \end{aligned}$$

and hence $d_{j+1} \le \cos \vartheta$. From this it follows that

$$|w_{j+1} - w_j|^2 \le |w_{j+1} - j|^2 \le 1 - d_{j+1} \cos \vartheta,$$

while as above

$$\begin{aligned} (1-b)^2 &\le (1-b)^2 |w_j - w_{j-1}|^2 \\ &\le \Big(|w_j - w_{j-1}| - |w_{j+1} - 2w_j + w_{j-1}|\Big)^2 \\ &\le |w_{j+1} - w_j|^2 \le 1 - d_{j+1} \cos \vartheta. \end{aligned}$$

Thus, $d_{j+1} < c$. If j is odd, then

$$\begin{aligned} d_{j+1} \cos \vartheta + 1 + d_j \cos \vartheta &\le |w_{j+1} - w_j| \\ &\le |w_{j+1} - 2w_j + w_{j-1}| + |w_j - w_{j-1}| \\ &\le (b+1)|w_j - w_{j-1}| \le b + 1, \end{aligned}$$

and $d_{j+1} < c$. □

4.13. *Remark.* When $\vartheta < \pi/2$, the arc α in Example 4.12 is a *simple zipper* in the sense of Thurston [T, §3]. Theorem 4.9 shows that the complements of all such quasiarcs are contained in $S \backslash \overline{T}$.

5. A COUNTEREXAMPLE

In [AG] we considered an alternative model for the Bers space T; namely, the set T_1 of logarithmic derivatives f''/f' of conformal mappings f of U into $\mathbb{C}$ with quasiconformal extensions to $\overline{\mathbb{C}}$ equipped with the norm

$$\|\phi\|_U^* = \sup_{z \in U} |\phi(z)|(1 - |z|^2). \tag{5.1}$$

Then $\pi(\phi) = \phi' - \frac{1}{2}\phi^2$ is a continuous map of T_1 into T, $\pi(\overline{T}_1) \subset \overline{T}$, and it is natural to ask if $\overline{T} = \pi(\overline{T}_1)$. We give an example here to show that this is not the case, thus completing the proof of Theorem 3.4 in [AG]. There we exhibited

a conformal mapping f with f''/f' in $\pi^{-1}(\overline{T})\backslash\overline{T}_1$ instead of $S_f = \pi(f''/f')$ in $\overline{T}\backslash\pi(\overline{T}_1)$, as asserted in Corollary 3.19. We are grateful to Dr. Y. Gatok for calling our attention to this oversight.

5.2. **Theorem.** *There exists a conformal mapping f of U into $\mathbb{C}$ for which S_f is in $\overline{T}\backslash\pi(\overline{T}_1)$.*

Our proof is based on the following preliminary calculation:

5.3. **Lemma.** *Suppose that $G = B \cup B'$ where B is an open disk, B' is an open disk or half-plane, and ∂B and $\partial B'$ meet in acute exterior angles. If h is a homeomorphism of $\overline{G}$ which maps G conformally into $\mathbb{C}$ and if*

$$\left|\frac{h''}{h'}(z)\right| \le c\,\mathrm{dist}(z, \partial G)^{-1} \tag{5.4}$$

for z in G, where $0 < c < 1$, then

$$\frac{\mathrm{dia}(h(C))}{|h(a) - h(b)|} \ge \frac{1-c}{16}\left(\frac{\mathrm{dia}(C)}{|a-b|}\right)^{1-c} - 1, \tag{5.5}$$

where $C = \partial B\backslash B'$ and $\{a, b\} = \partial B \cap \partial B'$.

Proof. By performing a preliminary similarity mapping, we may assume that $B = B(i, r)$ and that $a = -s$, $b = s$ where $s > 0$ and $s^2 + 1 = r^2$. Then for x in $I = (-s, s)$, $\mathrm{dist}(x, \partial G) = s - |x|$ and

$$\log\left|\frac{h'(x)}{h'(0)}\right| \le \left|\int_0^x \frac{h''}{h'}(\xi)\,d\xi\right| \le c\int_0^{|x|}(s-\xi)^{-1}\,d\xi = \log\left(1 - \frac{|x|}{s}\right)^{-c}$$

by (5.4). Hence

$$\mathrm{dia}(h(I)) \le \int_{-s}^{s}|h'(x)|\,dx \le \frac{2}{1-c}|h'(0)|s. \tag{5.6}$$

Similarly

$$\log\left|\frac{h'(0)}{h'(i)}\right| \le \left|\int_0^1 \frac{h''}{h'}(iy)\,dy\right| \le c\int_0^1 (y^2+s^2)^{-1/2}\,dy \le \log 2\left(\frac{r}{s}\right)^c$$

by (5.4) and we obtain

$$|h'(0)| \le 2\left(\frac{r}{s}\right)^c |h'(i)| \le 8\left(\frac{r}{s}\right)^c \frac{\mathrm{dist}(h(i), h(C))}{\mathrm{dist}(i, \partial G)} \tag{5.7}$$

from the Koebe distortion theorem. Since

$$\mathrm{dia}(h(C)) \ge \mathrm{dist}(h(i), h(C)) - \mathrm{dia}(h(I)),$$

(5.5) follows from (5.6) and (5.7). □

Proof for Theorem 5.2. Let $\{B_j\}$ be a sequence of open disks with centers in H and diameter 1 such that $B_j \cap \partial H$ is an open interval (a_j, b_j) and such that $|a_j - a_k| \ge 3$ for $j \ne k$ and

$$\lim_{j\to\infty}\frac{\mathrm{dia}(C_j)}{|a_j - b_j|} = \infty, \qquad C_j = \partial B_j\backslash L, \tag{5.8}$$

where L is the lower half-plane. Suppose next that f maps U conformally onto

$$D = \left(\bigcup_{j=1}^{\infty} B_j \right) \cup L .$$

We shall show that S_f is in $\overline{T} \backslash \pi(\overline{T}_1)$.

To show that S_f is in $\overline{T}$, fix $K > 1$, and for each j, let g_j denote a Möbius self-mapping of H for which $g_j(a_j) = 0$ and $g_j(b_j) = \infty$. Next set

$$h(re^{i\vartheta}) = \begin{cases} re^{i\vartheta/K} & \text{if } 0 \le \vartheta < \pi , \\ re^{i\vartheta} & \text{if } -\pi < \vartheta \le 0 , \end{cases}$$

and

$$g(z) = \begin{cases} g_j^{-1} \circ h \circ g_j(z) & \text{if } z \in B_j , \\ z & \text{if } z \in L . \end{cases}$$

Then g is a homeomorphism of D onto a quasidisk. We must show $g|B$ has a K_0-quasiconformal extension g_0 to $\overline{\mathbb{C}}$ for each open disk B in D where $K_0(K) \to 1$ as $K \to 1$. If $B \subset L$, we may take $g_0(z) = z$. If $B \cap B_j \cap H \neq \varnothing$, then $g_j(B)$ lies in some half-plane

$$G = \left\{ re^{i\vartheta} : |\vartheta - \phi| < \frac{\pi}{2} , 0 < r < \infty \right\} , \quad |\phi| < \frac{\pi}{2} ,$$

$h|G$ can be extended by means of a simple folding to a K-quasiconformal mapping h_0 of $\overline{\mathbb{C}}$, and $g_0 = g_j^{-1} \circ h_0 \circ g_j$ is the desired extension of $g|B$.

Suppose next that S_f is in $\pi(\overline{T}_1)$. Then $S_f = \pi(\phi''/\phi')$ where ϕ''/ϕ' is in $\overline{T}_1$ and there exists a conformal mapping ψ of U onto a quasidisk $D'' \subset \mathbb{C}$ with

$$\sup_{z \in D'} \left| \frac{h''}{h'}(z) \right| \rho_{D'}(z)^{-1} = \left\| \frac{h''}{h'} \right\|_{D'}^{*} = \left\| \frac{\phi''}{\phi'} - \frac{\psi''}{\psi'} \right\|_{U}^{*} \le c < 1 ,$$

where $D' = \phi(U)$ and $h = \psi \circ \phi^{-1}$. Hence

$$\left| \frac{h''}{h'}(z) \right| \le c \operatorname{dist}(z , \partial D')^{-1} \tag{5.9}$$

for z in D'. Since $S_f = S_\phi$, there exists a Möbius transformation g such that $\phi = g \circ f$ and hence $D' = g(D)$. Then, since ϕ has no poles in U, $g(D) \subset \mathbb{C}$, $g(L)$ is an open disk or half-plane and $g(B_j)$ is an open disk for $j \ge j_0$.

Now fix $j \ge j_0$. Then $\partial g(B_j)$ and $\partial g(L)$ form acute exterior angles and

$$\frac{\operatorname{dia}(g(C_j))}{|g(a_j) - g(b_j)|} \ge \frac{\operatorname{dia}(C_j)}{|a_j - b_j|} , \tag{5.10}$$

where a_j and b_j are the endpoints of C_j. Thus we obtain

$$\frac{\operatorname{dia}(h \circ g(C_j))}{|h \circ g(a_j) - h \circ g(b_j)|} \ge \frac{1-c}{16} \left(\frac{\operatorname{dia}(C_j)}{|a_j - b_j|} \right)^{1-c} - 1 \tag{5.11}$$

from (5.9), (5.10), and Lemma 5.3, with $B = g(B_j)$ and $B' = g(L)$. Then (5.8) and (5.11) imply that $\partial D'' = h \circ g(\partial D)$ is not a quasicircle, and we have a contradiction. □

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