CRICKETS, ZIPPERS,
AND THE BERS UNIVERSAL TEICHMÜLLER SPACE

K. ASTALA AND F. W. GEHRING

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Dedicated to Lipman Bers on his seventy-fifth birthday

Abstract. This paper contains a simple characterization of the conformal mappings $f$ of the unit disk $U$ whose Schwarzians lie in the closure of the Bers universal Teichmüller space. A second characterization is given and three examples are studied for the case where $f(U)$ is the complement of a quasiarc.

1. Introduction

Let $E$ denote the Banach space of functions $\phi$ analytic in the unit disk $U$ with norm

$$
\|\phi\|_U = \sup_{z \in U} |\phi(z)|(1 - |z|^2)^2,
$$

and let $S_f$ denote the Schwarzian derivative $(f''/f')' - \frac{1}{2} (f''/f')^2$ of a locally conformal mapping $f$ of a domain in the extended complex plane $\overline{\mathbb{C}}$.

In the 1960s, Bers showed how all Teichmüller spaces $T(G)$ of Fuchsian groups $G$ or Riemann surfaces $U/G$ could be embedded in a very natural manner into a universal Teichmüller space $T$, the subset of $E$ consisting of Schwarzian derivatives $S_f$ of conformal mappings $f$ of $U$ which have a quasiconformal extension to $\overline{\mathbb{C}}$. Besides providing new insight into the spaces $T(G)$, this embedding raised the interesting question of how $T$ is related to the subset $S$ of $E$ of Schwarzians $S_f$ of all conformal mappings $f$ of $U$. For example, Bers asked if $S = \overline{T}$ and, later, if $S$ is itself connected. It turns out that $S$ is closed in $E$, that $\text{int}S$ is equal to $T$ [G2] and hence is connected [EE], that $S \setminus \overline{T}$ is not empty [G3], and that $S$ contains isolated points and thus is not connected [T, As].

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In §2 of this paper we give a simple characterization of the conformal mappings of $U$ whose Schwarzians lie in $\mathcal{T}$; our approach is a modification of a method used in [AG]. We apply this result in §3 to obtain a second characterization for the case where $f(U) = \mathbb{C}\setminus \alpha$ and $\alpha$ is a quasiarcs. We consider three examples of this situation in §4 and show, in particular, that $S_f$ is not in $\mathcal{T}$ for a class of quasiarcs $\alpha$ which includes Thurston's simple zippers [T, §3]. We conclude in §5 with a remark on the relation between $\mathcal{T}$ and the corresponding space $T_1$, where $S_f$ is replaced by $f''/f'$.

2. Closure of $T$

Throughout this paper, all quasiconformal mappings are assumed to be sense preserving. Given a simply connected domain $D$ of hyperbolic type in $\mathbb{C}$, we let $\rho_D$ denote the density of the hyperbolic metric with curvature $-4$ in $D$. Next, if $f$ is conformal in $D$, we set

$$\|S_f\|_D = \sup_{z \in D} |S_f(z)|\rho_D(z)^{-2}. $$

This is the counterpart of (1.1) for the domain $D$.

2.2. Theorem. Suppose that $f$ is a conformal mapping of $U$ onto $D$. Then $S_f$ is in $\mathcal{T}$ if and only if for each $K > 1$ there exists a homeomorphism $g$ of $D$ onto a quasidisk such that for each disk $B$ in $D$, $g|B$ has a $K$-quasiconformal extension to $\mathbb{C}$.

Proof. Suppose that $S_f$ is in $\mathcal{T}$ and that $B$ is a disk in $D$. For each $K > 1$, we can choose a conformal mapping $\phi$ of $U$ onto a quasidisk with

$$\|S_f\|_B \leq \|S_f\|_D = \|S_f - S_{\phi}\|_U \leq 2\frac{K - 1}{K + 1},$$

where $g = \phi \circ f^{-1}$. Then by [L, Theorems II.4.1 and II.5.1], $g|B$ has a quasiconformal extension to $\mathbb{C}$ with complex dilatation $\mu_g$ where

$$\|\mu_g\|_{L^\infty} \leq \frac{1}{2}\|S_f\|_D \leq \frac{K - 1}{K + 1},$$

hence establishing the necessity.

For the sufficiency, fix $0 < \varepsilon < 1$ and $c \geq 1$, choose $K > 1$ so that

$$\varepsilon = 96c^2 \frac{K^2 - 1}{K^2 + 1},$$

and let $g$ be the homeomorphism corresponding to $K$ in the hypotheses. Then $g$ is $K$-quasiconformal in a neighborhood of each point of $D\setminus \{\infty\}$ and hence in $D$; by the measurable Riemann mapping theorem, there exists an $h$, $K$-quasiconformal in $\mathbb{C}$, such that $\psi = h \circ g$ is conformal in $D$.

Fix $z_0$ in $D\setminus \{\infty\}$ and let $B = B(z_0, r)$ be the disk with center $z_0$ and radius $r$ where $cr = \text{dist}(z_0, \partial D)$. If $g_B$ is the extension of $g|B$, then
\[ \psi_B = h \circ g_B \] is \( K^2 \)-quasiconformal in \( \overline{\mathbb{C}} \), \( \psi_B \) agrees with \( \psi \) in \( B \), and

\[ |S_\psi(z_0)|^2 \leq \|S_\psi\|^2_B \leq 6 \frac{K^2 - 1}{K^2 + 1} \frac{e \epsilon^{-2}}{16} \]

[L, Theorem II.3.2]. The Koebe distortion theorem implies that

\[ \rho_{D}(z_0)^{-1} \leq 4 \text{dist}(z_0, \partial D) \leq 4c \rho_B(z_0)^{-1} \]

and hence that

\[ |S_\psi(z_0)|^2 \rho_D(z_0)^{-2} \leq 16c^2 |S_\psi(z_0)| \rho_B(z_0)^{-2} \leq \epsilon. \]

Thus

\[ \|S_f - S_\psi\|_U = \|S_\psi\|_D \leq \epsilon \]

where \( \phi = \psi \circ f \), \( \phi(U) = \psi(D) \) is a quasidisk and \( S_f \) is in \( \mathcal{F} \). \( \square \)

2.3. Remark. The above proof shows that \( S_f \) is in \( \mathcal{F} \) if for some fixed \( c \geq 1 \) and each \( K > 1 \) there exists a homeomorphism \( g \) of \( D \) onto a quasidisk such that \( g|B \) has a \( K \)-quasiconformal extension to \( \overline{\mathbb{C}} \) for all \( B = B(z_0, r) \) in \( D \) with \( cr \leq \text{dist}(z_0, \partial D) \).

For convenience of notation we let \( S, T \), and \( \mathcal{F} \) also denote the family of images \( f(U) \) when \( S_f \) is in \( S \), \( T \), and \( \mathcal{F} \), respectively. With this convention, we see that if \( g \) is a Möbius transformation, then \( D \) is in \( \mathcal{F} \) whenever \( g(D) \) is in \( \mathcal{F} \). We have the following useful extension of this observation:

2.4. Corollary. A domain \( D \) is in \( \mathcal{F} \) if for each \( K > 1 \) there exists a \( K \)-quasiconformal self-mapping \( g \) of \( \overline{\mathbb{C}} \) with \( g(D) \) in \( \mathcal{F} \).

Proof. By a distortion theorem for quasiconformal mappings, we can choose a constant \( c \geq 1 \) with the following property: If \( B = B(z_0, r) \) is in \( D \subset \mathbb{C} \) with \( cr \leq \text{dist}(z_0, \partial D) \) and if \( g \) is a \( K \)-quasiconformal mapping of \( D \) onto \( D' \subset \mathbb{C} \) with \( 1 \leq K \leq 2 \), then \( g(B) \) lies inside a disk \( B' \) in \( D' \). See [G1, p. 383] or [V1, Theorem 18.1].

Suppose \( g \) is \( K \)-quasiconformal in \( \overline{\mathbb{C}} \) with \( 1 < K \leq 2 \) and \( D' = g(D) \) in \( \mathcal{F} \); as noted above, we may assume that \( D \) and \( D' \) lie in \( \mathbb{C} \). By Theorem 2.2, there exists a homeomorphism \( h \) of \( D' \) onto a quasidisk \( D'' \) such that \( h|B' \) has a \( K \)-quasiconformal extension to \( \overline{\mathbb{C}} \) for each disk \( B' \) in \( D' \). Then \( h \circ g(D) = D'' \), and \( h \circ g|B \) has a \( K^2 \)-quasiconformal extension to \( \overline{\mathbb{C}} \) for \( B = B(z_0, r) \) in \( D \) with \( cr \leq \text{dist}(z_0, \partial D) \). \( \square \)

3. The complement of a quasiarc

Most of the known domains in \( S \setminus \mathcal{F} \) are of the form \( D = \mathbb{C} \setminus \alpha \), where \( \alpha \) is a quasiarc, i.e., the image of the closed positive half \( I = [0, \infty) \) of the extended real axis \( \mathbb{R} \) under a quasiconformal mapping of \( \overline{\mathbb{C}} \). (cf. [As, G3, T]. But see also [F].) We use Theorem 2.2 to derive the following criterion for the complement of a quasiarc to be in \( \mathcal{F} \).
3.1. **Cricket Theorem.** Suppose that $\alpha$ is a subarc of a quasicircle $\gamma$ in $\mathbb{C}$. Then $D = \mathbb{C} \setminus \alpha$ is in $T$ if and only if for each $K > 1$ there exists a sense preserving $K$-quasiconformal self-mapping $h$ of $\mathbb{C}$ such that $h(z) = z$ in $\gamma \setminus \alpha$ and $h(\alpha) \cup \alpha$ is a quasicircle.

3.2. **Remark.** If we think of $\gamma$ as a band of spring steel, Theorem 3.1 says that $D = \mathbb{C} \setminus \alpha$ is in $T$ if and only if the part corresponding to $\alpha$ can be snapped by small deformations into positions which differ markedly from the original, while the remainder of the band remains fixed. This similarity to a metal toy cricket is the reason for the Theorem's name.

**Proof of the necessity in Theorem 3.1.** Let $G_1$ and $G_2$ denote the components of $\mathbb{C} \setminus \gamma$. Then there exists a constant $a = a(\gamma) > 0$ such that each $f$ conformal in $G_j$ with $\|S_f\|_{G_j} < a$ has a quasiconformal extension $g_j$ to $\mathbb{C}$ with

\[
\|S_f\|_{G_j} \leq \frac{1}{a} \|S_f\|_{G_j} \tag{3.3}\]

[L, Theorem II.4.1].

Fix $K > 1$, and choose $0 < \varepsilon < 1$ so that $(1 + \varepsilon)/(1 - \varepsilon) \leq \sqrt{K}$. Because $D$ is in $T$, there exists a conformal mapping $f$ of $D$ onto a quasidisk $D'$ with $\|S_f\|_{G_j} \leq \|S_f\|_D < ae$. By (3.3), $f|G_j$ has a quasiconformal extension $g_j$ to $\mathbb{C}$ with maximal dilatation $K(g_j) \leq \sqrt{K}$. Then $h = g_2^{-1} \circ g_1$ is $K$-quasiconformal in $\mathbb{C}$, $h$ fixes each point of $\gamma \setminus \alpha = \beta$, and $g_1(\alpha) \cup g_2(\alpha) = \partial D'$. Hence $h(\alpha) \cup \alpha = g_2^{-1}(\partial D')$ is a quasicircle. \(\square\)

The proof for the sufficiency in Theorem 3.1 requires several preliminary results.

3.4. **Lemma.** For each $0 < a < \infty$, there exists $0 < b = b(a) < \infty$ with the following property: If $g$ is $K$-quasiconformal in $\mathbb{C}$, with $1 \leq K \leq e^{a/b}$, and if $g$ fixes $z_1, z_2, \infty$, then

\[
|g(z) - z| \leq b|z_1 - z_2| \log K \tag{3.5}\]

for $z$ in $B(z_1, a|z_1 - z_2|)$.

**Proof.** The argument is similar to that for Lemma 7 in [G3]. Let $\rho_G$ and $h_G$ denote the hyperbolic density and metric with curvature $-4$ in $G = \mathbb{C} \setminus \{0, 1, \infty\}$ and choose $b$ so that

\[
1 = 2b \inf\{\rho_G(z): z \in G \cap \overline{B}(0, 2a)\}.
\]

Then, by a theorem of Teichmüller [Ah, pp. 53–61],

\[
h_G(g(z), z) \leq \frac{1}{2} \log K
\]

for each $K$-quasiconformal mapping $g$ of $\mathbb{C}$ which fixes $0, 1, \infty$. 

Suppose that $g$ satisfies the above hypotheses with $z_1 = 0$ and $z_2 = 1$. Fix $z \in G \cap B(0, a)$, and let $\gamma$ denote the hyperbolic geodesic joining $z$ and $g(z)$ in $G$ and $\ell(\gamma)$ the euclidean length of $\gamma$. Then $\gamma \subset B(0, 2a)$ since otherwise

$$a < \ell(\gamma \cap \overline{B}(0, 2a)) \leq 2b \int_\gamma \rho_G \, ds = 2b h_G(g(z), z) \leq b \log K,$$

contradicting our choice of $K$. Thus

$$|g(z) - z| \leq \ell(\gamma) \leq 2b \int_\gamma \rho_G \, ds \leq b \log K,$$

and we obtain (3.5) for the case where $z_1 = 0$ and $z_2 = 1$. The general case follows from applying what was proved above to

$$h(w) = \frac{g(z) - z_1}{z_2 - z_1}, \quad z = w(z_2 - z_1) + z_1. \quad \Box$$

A continuous map $g$ of $E \subset \mathbb{C}$ into $\mathbb{C}$ is $s$-quasisymmetric [V2], $0 < s < \infty$, if, for each three points $z, z', z''$ in $E$,

$$s < \frac{|z'-z|}{|z''-z|} \leq \frac{1}{s} \quad \text{implies that} \quad \frac{|g(z') - g(z)|}{|g(z'') - g(z)|} \leq \frac{|z'-z|}{|z''-z|} + s.$$

3.6. Lemma. For each $s > 0$ there exist $t > 0$ and $K > 1$ with the following property: If $g$ is $K$-quasiconformal and satisfies $|g(z) - z| \leq tr$ in $B(z_0, 2r)$, then $g$ is $s$-quasisymmetric in $B(z_0, r)$.

Proof. Since the composition of an $s$-quasisymmetric map with a euclidean similarity is $s$-quasisymmetric, we may assume $z_0 = 0$ and $r = 1$.

Suppose that the conclusion of Lemma 3.6 does not hold for some $s > 0$. Then for each integer $j$ there exists a homeomorphism $g_j$ which is $K_j$-quasiconformal and satisfies $|g_j(z) - z| \leq t_j$ in $B(0, 2)$, where $K_j \to 1$ and $t_j \to 0$. In addition, there exist points $z_j, z'_j, z''_j$ in $B(0, 1)$ such that

$$s < \frac{|z'_j - z_j|}{|z''_j - z_j|} \leq \frac{1}{s} \quad \text{and} \quad \frac{|w'_j - w_j|}{|w''_j - w_j|} - \frac{|z'_j - z_j|}{|z''_j - z_j|} > s,$$

where $w_j = g_j(z_j), \quad w'_j = g_j(z'_j), \quad w''_j = g_j(z''_j)$. By passing to a subsequence we may assume that $z_j, \quad w_j \to z_0, \quad z'_j, \quad w'_j \to z'_0, \quad z''_j, \quad w''_j \to z''_0$. Condition (3.7) then implies that $z_0 = z'_0 = z''_0$.

Let $h_j = \psi_j \circ g \circ \phi_j^{-1}$, where $\phi_j$ and $\psi_j$ are euclidean similarities with

$$\phi_j(z_j) = 0, \quad \phi_j(z'_j) = 1, \quad \psi_j(w_j) = 0, \quad \psi_j(w''_j) = 1.$$

Then $h_j$ is $K_j$-quasiconformal in

$$B_j = \phi_j(B(0, 2)) \supset B(0, r_j), \quad r_j = |z'_j - z_j|^{-1}. \quad \Box$$
Since \( h_j \) fixes 0, 1 and omits \( \infty \) in \( B_j \), by passing to a second subsequence we may assume that \( |h_j(z)| \to |z| \) locally uniformly in \( \mathbb{C} \). Then \( |\phi_j(z')| \) is bounded in \( j \) by the first part of (3.7) and
\[
\frac{|w'_j - w_j|}{|w''_j - w_j|} = \frac{|z_j' - z_j|}{|z''_j - z_j|} = |h_j(\phi_j(z'))| - |\phi_j(z')| \to 0,
\]
which contradicts the second part of (3.7). \( \square \)

3.8. Lemma. For each \( K > 1 \), there exists \( s > 0 \) with the following property: If \( g \) is an \( s \)-quasisymmetric homeomorphism of a closed disk \( B \), then \( g|B \) has a homeomorphic extension to \( \bar{\mathbb{C}} \) which is \( K \)-quasiconformal in \( \bar{\mathbb{C}} \setminus B \).

Proof. This follows from [TV, Theorems 5.23 and 2.6]. \( \square \)

Proof of sufficiency in Theorem 3.1. By performing a preliminary Möbius transformation, we may assume that \( \alpha \) has 0 and \( \infty \) as its endpoints. Next, by hypothesis, there exists for each \( K > 1 \) a \( K \)-quasiconformal self-mapping \( h \) of \( \mathbb{C} \) such that \( h(z) = z \) for \( z \) in \( \beta = \gamma \setminus \alpha \), and \( h(\alpha) \cup \alpha \) is a quasicircle. Hence \( h \) fixes 0 and \( \infty \),
\[ h(\alpha) \cap \gamma = h(\alpha) \cap \alpha = \{0, \infty\} \]
and we can label the components \( G_1 \) and \( G_2 \) of \( \mathbb{C} \setminus \gamma \) so that \( h(\alpha) \subset G_2 \cup \{0, \infty\} \). Then the fact that \( h \) is sense preserving and the identity on \( \gamma \setminus \alpha \) implies that \( h(G_2) \subset G_2 \). Hence
\[ g(z) = \begin{cases} z & \text{if } z \in G_1 \cup \beta, \\ h(z) & \text{if } z \in G_2 \end{cases} \]
defines a homeomorphism of \( D = G_1 \cup \beta \cup G_2 \) onto \( D' = G_1 \cup \beta \cup h(G_2) \), and \( \partial D' = h(\alpha) \cup \alpha \) is a quasicircle.

To complete the proof, it suffices to show that for each \( B = B(z_0, r) \) in \( D \) with \( 3r \leq \text{dist}(z_0, \partial D) \), \( g|B \) has a \( K_0 \)-quasiconformal extension \( g_0 \) to \( \bar{\mathbb{C}} \) where \( K_0 = K_0(K) \to 1 \) as \( K \to 1 \). If \( B \subset G_1 \), we may take \( g_0(z) = z \) and \( K_0 = 1 \); if \( B \subset G_2 \), we may choose \( g_0(z) = h(z) \) and \( K_0 = K \).

It remains to consider the case where there exists a point \( z_1 \in B \cap \beta \). Since \( \beta \) is unbounded, we can choose \( z_2 \in \beta \) with \( |z_1 - z_2| = r \). Then \( h \) is \( K \)-quasiconformal in \( \mathbb{C} \) and fixes \( z_1, z_2, \infty \). Hence, by Lemma 3.4 applied to \( h \) in \( B(z_1, 3r) \), there exists an absolute constant \( b > 0 \) such that
\[ |g(z) - z| \leq |h(z) - z| \leq t r, \quad t = b \log K \]
if \( z \in B(z_0, 2r) \) and \( 1 \leq K \leq e^{3/b} \). Since \( g \) is \( K \)-quasiconformal in \( B(z_0, 2r) \), Lemma 3.6 implies that \( g \) is \( s \)-quasisymmetric in \( \overline{B(z_0, r)} \), where \( s = s(K) \to 0 \) as \( K \to 1 \). Then by Lemma 3.8, \( g|B \) has a \( K_0 \)-quasiconformal extension to \( \mathbb{C} \) where \( K_0 = K_0(s) \to 1 \) as \( s \to 0 \). \( \square \)

4. Examples

We apply the Cricket Theorem to give two classes of quasicrds \( \alpha \) for which \( D = \mathbb{C} \setminus \alpha \) is in \( \mathcal{T} \) and a third for which \( D = \mathbb{C} \setminus \alpha \) is not in \( \mathcal{T} \).
A quasicircle $\alpha$ is \textit{asymptotically conformal} if, for each $K > 1$, $\alpha = g(I)$, where $g$ is a quasiconformal in $\mathbb{C}$ and $K$-quasiconformal in a neighborhood $V$ of $I = [0, \infty]$. If $\alpha \subset \mathbb{C}$ is asymptotically conformal, then

$$\frac{|z_1 - z_2| + |z_2 - z_3|}{|z_1 - z_3|} \to 1 \quad \text{as} \quad |z_1 - z_3| \to 0,$$

for each ordered triple of points $z_1, z_2, z_3$ in $\alpha$.

4.1. \textbf{Theorem.} If $\alpha$ is an asymptotically conformal quasicircle, then $D = \mathbb{C} \setminus \alpha$ is in $\mathcal{T}$.

\textit{Proof.} Fix $K > 1$. Then there exists $g$ quasiconformal in $\mathbb{C}$ with $g$ $K$-quasiconformal in a neighborhood $V$ of $I$ and $g(I) = \alpha$. Choose $0 < a < \pi$ and $0 < b < a$ so that

$$W = \{ z = re^{i\theta} : 0 \leq r \leq \infty, |\theta| \leq a \} \subset V, \quad \frac{a}{a - b} = K$$

and set

$$f(re^{i\theta}) = re^{i\phi(\theta)}, \quad f(\infty) = \infty,$$

where $\phi$ is piecewise linear and increasing in $[-\pi, \pi]$ with $\phi(\pm \pi) = \pm \pi$, $\phi(\pm a) = \pm a$ and $\phi(0) = b$. Then $f$ is a $K$-quasiconformal self-mapping of $\mathbb{C}$, $f(W) = W$ and $f(z) = z$ in $\mathbb{C} \setminus W$.

Let $h = g \circ f \circ g^{-1}$. Then $h$ is $K^3$-quasiconformal in $g(V)$ and each point of $\mathbb{C} \setminus g(W)$ has a neighborhood in which $h(z) = z$. Thus $h$ is $K^3$-quasiconformal in $\mathbb{C}$, $h(\alpha) \cup \alpha$ is a quasicircle and $D$ is in $\mathcal{T}$ by Theorem 3.1 with $\gamma = g(R)$ and $R$ the real axis. \(\Box\)

A quasicircle $\alpha$ is a \textit{graph} if for each line $\lambda$ parallel to some fixed line, the set $\alpha \cap \lambda$ is either connected or empty.

4.3. \textbf{Theorem.} If $\alpha$ is a quasicircle which is a graph, then $D = \mathbb{C} \setminus \alpha$ is in $\mathcal{T}$.

\textit{Proof.} We consider here only the case where $\alpha$ is unbounded. Then, by means of a preliminary similarity mapping, we may assume that $\alpha \cap \lambda$ is connected or empty for each line $\lambda$ parallel to the $y$-axis, that $\alpha$ has endpoints 0 and $\infty$, and that $\alpha$ does not meet the left half-plane.

Suppose that there exists a constant $1 \leq m < \infty$ such that

$$|z_1 - z_2| \leq m|\Re(z_1) - \Re(z_2)|$$

for each $z_1, z_2$ in $\alpha \cap \mathbb{C}$. Given $K > 1$, choose $t$ so that $$(1 + t)/(1 - t) = K,$$

and let

$$h(z) = \begin{cases} z + it\Re(z) & \text{if } 0 \leq \Re(z) < \infty, \\ z & \text{if } -\infty < \Re(z) \leq 0, \end{cases}$$

and $h(\infty) = \infty$. Then $h$ is $K$-quasiconformal in $\mathbb{C}$ and (4.4) implies that $h(\alpha) \cup \alpha$ is a quasicircle. Hence $D = \mathbb{C} \setminus \alpha$ is in $\mathcal{T}$ by Theorem 3.1.

For the general case, fix $K > 1$ and let $\beta = (-\infty, 0)$ denote the negative half of the real axis. Then $\gamma = \alpha \cup \beta$ is a quasicircle. Let $f$ map the upper half-plane
$H$ conformally onto the upper component $G$ of $\mathbb{C} \setminus \gamma$ so that its homeomorphic extension to $H$ fixes $\infty$. Because $\alpha$ is a graph, $\{z + it: z \in G\} \subset G$ for $t > 0$. Hence, if we let $\phi$ map $H$ conformally onto $U$ so that $0$ and $\infty$ correspond to $-1$ and $1$, respectively, we can apply [HS, pp. 313–314] to $\psi = f \circ \phi^{-1}$ to conclude that $\text{Re}(f'(z)) \geq 0$ in $H$.

Now set

$$g(z) = \int_{i}^{z} f'(\zeta) d\zeta, \quad h = g \circ f^{-1},$$

for $0 < t < 1$. Then

$$||S_h||_G = ||S_f - S_g||_H \leq (1 - t) \sup_{z \in H} \left( \left| S_f(z) \right| + \frac{t}{2} \left| \frac{f'''(z)}{f'(z)} \right| \right) \rho_H(z)^{-2} \leq 24(1 - t),$$

$h$ has a $K$-quasiconformal extension to $\mathbb{C}$ for $t$ near $1$, and

$$|g'(z)| \leq m \text{Re}(g'(z)), \quad m = \sec(\pi t/2),$$

in $H$. If $-\infty < \xi_1 < \xi_2 < \infty$ and $0 < \eta < \infty$, then

$$|z_1 - z_2| \leq \int_{\xi_1}^{\xi_2} |g'(\xi + i\eta)| d\xi \leq m \int_{\xi_1}^{\xi_2} \text{Re}(g'(\xi + i\eta)) d\xi = m|\text{Re}(z_1) - \text{Re}(z_2)|,$$

where $z_j = g(\xi_j + i\eta)$, and letting $\eta \to 0$ yields (4.4) for $z_1, z_2$ in $h(\alpha) \cap \mathbb{C}$. Thus $h(\mathbb{C} \setminus \alpha)$ is in $\mathcal{T}$ and hence so is $\mathbb{C} \setminus \alpha$ by Corollary 2.4. 

Theorem 4.3 allows us to characterize those quasiarcs, which are invariant with respect to a homothety $f(z) = az$, with complements in $\mathcal{T}$.

4.5. **Corollary.** Suppose that $\alpha$ is a quasiarc, that $0$ is an interior point of $\alpha$, and that $f(\alpha) \subset \alpha$ for some $a$ in $B(0, 1) \setminus \{0\}$. Then $D = \mathbb{C} \setminus \alpha$ is in $\mathcal{T}$ if and only if $\alpha$ is a graph.

**Proof.** The necessity follows from [As, Theorem 4.1] and the sufficiency from Theorem 4.3. 

The consideration of quasiarcs that are invariant with respect to a translation $f(z) = z + a$ leads us to the notion of an interlocking sequence. Given any triple of points $z_{j-1}, z_j, z_{j+1}$ in $\mathbb{C}$, we let

$$r(z_j) = \frac{z_{j+1} - z_j}{z_{j-1} - z_j}.$$

Suppose next that $\alpha$ is a subarc of a quasicircle $\gamma$ which has $0$ and $\infty$ as its endpoints. A sequence of distinct points $z_j$ in $\alpha$ is interlocking if there exist constants $0 < a, b < \infty$, disjoint neighborhoods $V_j$ of $z_j$, and a component $G$ of $\mathbb{C} \setminus \gamma$ with the following properties:

$$|r(z_j)| \leq a \text{ for } j \geq 2.$$
(4.7) \[ \begin{cases} \text{If } w_{j-1} \in G \cap V_{j-1}, \ w_j \in G \cap V_j, \ w_{j+1} \in G, \text{ and } |r(z_j) - r(w_j)| \leq b \\ \text{for some } j \geq 2, \text{ then } w_{j+1} \in V_{j+1}. \end{cases} \]

(4.8) \[ \text{dia}(V_j)/|z_j| \to 0 \quad \text{as } j \to \infty. \]

Here \( \text{dia}(V_j) \) denotes the euclidean diameter of \( V_j \).

4.9. **Theorem.** If \( \alpha \) is a quasicircle that contains an interlocking sequence, then \( D = \overline{\mathbb{C}} \setminus \alpha \) is not in \( \overline{T} \).

**Proof.** By hypothesis there exist points \( z_j \) in \( \alpha \), neighborhoods \( V_j \) of \( z_j \), and a component \( G \) of \( \overline{\mathbb{C}} \setminus \gamma \) which satisfy (4.6) through (4.8); here \( \gamma \) is any quasicircle which contains \( \alpha \).

Suppose that \( D = \overline{\mathbb{C}} \setminus \alpha \) is in \( \overline{T} \). Then for each \( K > 1 \) there exists a \( K \)-quasiconformal self-mapping \( h \) of \( \overline{\mathbb{C}} \) with \( h(z_j) = z_j \) in \( \gamma \setminus \alpha \) and \( h(\alpha) \cup \alpha \) a quasicircle. By replacing \( h \) by \( h^{-1} \) if necessary, we may assume that

\[ w_j = h(z_j) \in h(\alpha \setminus \{0, \infty\}) \subset G, \quad j \geq 1. \]

Next, by Lemma 3.4, we can choose \( K > 1 \) so that \( w_1 \in V_1 \) and \( w_2 \in V_2 \) and so that each \( K \)-quasiconformal self-mapping \( g \) of \( \overline{\mathbb{C}} \) which fixes 0, 1, \( \infty \) satisfies

(4.10) \[ |g(w) - w| \leq b, \]

for \( w \) in \( B(0, a) \). Then (4.6) and (4.10) applied to

\[ g(w) = \frac{h(z) - w_j}{w_j - w_j}, \quad z = w(z_{j-1} - z_j) + z_j, \]

yield

\[ |r(w_j) - r(z_j)| = |g(r(z_j)) - r(z_j)| \leq b, \quad j \geq 2. \]

Since \( w_1 \in V_1 \) and \( w_2 \in V_2 \), (4.7) implies that \( w_j \in V_j \) for all \( j \) and hence that

(4.11) \[ \lim_{j \to \infty} \frac{|h(z_j) - z_j|}{|z_j|} = 0. \]

Because \( h(z_j) \) and \( z_j \) separate 0 and \( \infty \) in \( h(\alpha) \cup \alpha \), (4.11) then implies that \( h(\alpha) \cup \alpha \) is not a quasicircle. \( \square \)

4.12. **Example.** For \( \pi/3 < \vartheta < 2\pi/3 \) and \( j \geq 1 \), let \( \alpha_j \) denote the polygonal arc formed by joining successively the points \( 2j - 2, 2j - 1, 2j - 1 + e^{i(\pi - \vartheta)}, 2j + e^{i\vartheta}, 2j \) with linear segments. Next let

\[ \alpha = \left( \bigcup_{j=1}^{\infty} \alpha_j \right) \cup \{\infty\}, \]

and let \( \beta = (-\infty, 0) \) be the negative half of the real axis. Then \( \gamma = \alpha \cup \beta \) is a quasicircle. Moreover, \( \alpha \) contains an interlocking sequence when \( \vartheta < \pi/2 \), \( \alpha \) is a graph when \( \vartheta \geq \pi/2 \), and hence \( D = \overline{\mathbb{C}} \setminus \alpha \) is in \( \overline{T} \) if and only if \( \vartheta \geq \pi/2 \).
Proof. Suppose that $\theta < \pi/2$. We shall show that the points $z_j = j$ are interlocking in $\alpha$ with

$$a = 1, \quad b = \frac{\cos^2 \theta}{8}, \quad c = \frac{2b}{\cos \theta}, \quad V_j = B(j, c)$$

and $G$ the upper component of $\mathbb{C} \setminus \gamma$. Since (4.6) and (4.8) hold trivially, we need only prove that, if $w_{j-1}, w_j, w_{j+1}$ satisfy the hypotheses in (4.7), then $d_{j+1} < c$ where $d_k = |w_k - k|$.

If $j$ is even, then

$$d_{j+1} - 2d_j - d_{j-1} \leq |w_{j+1} - 2w_j + w_{j-1}|$$

$$= |r(w_j) - r(z_j)| |w_j - w_{j-1}|$$

$$\leq b|w_j - w_{j-1}| \leq b(d_j + 1 + d_{j-1}),$$

and hence $d_{j+1} \leq \cos \theta$. From this it follows that

$$|w_{j+1} - w_j|^2 \leq |w_{j+1} - j|^2 \leq 1 - d_{j+1} \cos \theta,$$

while as above

$$(1 - b)^2 \leq (1 - b)^2 |w_j - w_{j-1}|^2$$

$$\leq \left( |w_j - w_{j-1}| - |w_{j+1} - 2w_j + w_{j-1}| \right)^2$$

$$\leq |w_{j+1} - w_j|^2 \leq 1 - d_{j+1} \cos \theta.$$}

Thus, $d_{j+1} < c$. If $j$ is odd, then

$$d_{j+1} \cos \theta + 1 + d_j \cos \theta \leq |w_{j+1} - w_j|$$

$$\leq |w_{j+1} - 2w_j + w_{j-1}| + |w_j - w_{j-1}|$$

$$\leq (b + 1)|w_j - w_{j-1}| \leq b + 1,$$

and $d_{j+1} < c$. $\square$

4.13. Remark. When $\theta < \pi/2$, the arc $\alpha$ in Example 4.12 is a simple zipper in the sense of Thurston [T, §3]. Theorem 4.9 shows that the complements of all such quasiarcs are contained in $S \setminus \overline{T}$.

5. A COUNTEREXAMPLE

In [AG] we considered an alternative model for the Bers space $T$; namely, the set $T_1$ of logarithmic derivatives $f''/f'$ of conformal mappings $f$ of $U$ into $\mathbb{C}$ with quasiconformal extensions to $\overline{\mathbb{C}}$ equipped with the norm

$$\|\phi\|_{T_1} = \sup_{z \in U} |\phi(z)|(1 - |z|^2).$$

Then $\pi(\phi) = \phi' - \frac{1}{2} \phi^2$ is a continuous map of $T_1$ into $T$, $\pi(\overline{T_1}) \subset \overline{T}$, and it is natural to ask if $\overline{T} = \pi(\overline{T_1})$. We give an example here to show that this is not the case, thus completing the proof of Theorem 3.4 in [AG]. There we exhibited
a conformal mapping $f$ with $f''/f'$ in $\pi^{-1}(T) \setminus \overline{T_1}$ instead of $S_f = \pi(f''/f')$ in $T \setminus \pi(T_1)$, as asserted in Corollary 3.19. We are grateful to Dr. Y. Gatok for calling our attention to this oversight.

5.2. **Theorem.** There exists a conformal mapping $f$ of $U$ into $\mathbb{C}$ for which $S_f$ is in $T \setminus \pi(T_1)$.

Our proof is based on the following preliminary calculation:

5.3. **Lemma.** Suppose that $G = B \cup B'$ where $B$ is an open disk, $B'$ is an open disk or half-plane, and $\partial B$ and $\partial B'$ meet in acute exterior angles. If $h$ is a homeomorphism of $G$ which maps $G$ conformally into $\mathbb{C}$ and if

\[(5.4) \quad \left| \frac{h''}{h'}(z) \right| \leq c \text{dist}(z, \partial G)^{-1} \]

for $z$ in $G$, where $0 < c < 1$, then

\[(5.5) \quad \frac{\text{dia}(h(C))}{|h(a) - h(b)|} \geq \frac{1 - c}{16} \left( \frac{\text{dia}(C)}{|a - b|} \right)^{1-c} - 1, \]

where $C = \partial B \setminus B'$ and $\{a, b\} = \partial B \cap \partial B'$.

**Proof.** By performing a preliminary similarity mapping, we may assume that $B = B(i, r)$ and that $a = -s, b = s$ where $s > 0$ and $s^2 + 1 = r^2$. Then for $x$ in $I = (-s, s)$, dist$(x, \partial G) = s - |x|$ and

\[\log \left| \frac{h'(x)}{h'(0)} \right| \leq \int_0^x \frac{h''(\xi)}{h'(-\xi)} d\xi \leq c \int_0^{s(x)} (s - \xi)^{-1} d\xi = \log \left( 1 - \frac{|x|}{s} \right)^{-c} \]

by (5.4). Hence

\[(5.6) \quad \text{dia}(h(I)) \leq \int_{-s}^s |h'(x)| \, dx \leq \frac{2}{1-c} |h'(0)| s. \]

Similarly

\[\log \left| \frac{h'(0)}{h'(i)} \right| \leq \int_0^1 \frac{h''(iy)}{h'(iy)} \, dy \leq c \int_0^1 (y^2 + s^2)^{-1/2} \, dy \leq \log \left( \frac{r}{s} \right)^c \]

by (5.4) and we obtain

\[(5.7) \quad \left| h'(0) \right| \leq 2 \left( \frac{r}{s} \right)^c \left| h'(i) \right| \leq 8 \left( \frac{r}{s} \right)^c \frac{\text{dist}(h(i), h(C))}{\text{dist}(i, \partial G)} \]

from the Koebe distortion theorem. Since

\[\text{dia}(h(C)) \geq \text{dist}(h(i), h(C)) - \text{dia}(h(I)), \]

(5.5) follows from (5.6) and (5.7). □

**Proof for Theorem 5.2.** Let $\{B_j\}$ be a sequence of open disks with centers in $H$ and diameter 1 such that $B_j \cap \partial H$ is an open interval $(a_j, b_j)$ and such that $|a_j - a_k| \geq 3$ for $j \neq k$ and

\[(5.8) \quad \lim_{j \to \infty} \frac{\text{dia}(C_j)}{|a_j - b_j|} = \infty, \quad C_j = \partial B_j \setminus L, \]

where $L$ is the limit of $\partial B_j$. Since $B_j \cap \partial H$ is an open interval $(a_j, b_j)$ and $|a_j - a_k| \geq 3$ for $j \neq k$, we have

\[\text{dia}(C_j) = \frac{2}{1-c} |h'(0)| s. \]

Then

\[\log \left| \frac{h'(0)}{h'(i)} \right| \leq \int_0^1 \frac{h''(iy)}{h'(iy)} \, dy \leq c \int_0^1 (y^2 + s^2)^{-1/2} \, dy \leq \log \left( \frac{r}{s} \right)^c \]

by (5.4) and we obtain

\[\left| h'(0) \right| \leq 2 \left( \frac{r}{s} \right)^c \left| h'(i) \right| \leq 8 \left( \frac{r}{s} \right)^c \frac{\text{dist}(h(i), h(C))}{\text{dist}(i, \partial G)} \]

from the Koebe distortion theorem. Since

\[\text{dia}(h(C)) \geq \text{dist}(h(i), h(C)) - \text{dia}(h(I)), \]

(5.5) follows from (5.6) and (5.7). □
where \( L \) is the lower half-plane. Suppose next that \( f \) maps \( U \) conformally onto
\[
D = \left( \bigcup_{j=1}^{\infty} B_j \right) \cup L.
\]
We shall show that \( S_f \) is in \( \overline{T \setminus \pi(T_1)} \).

To show that \( S_f \) is in \( \overline{T} \), fix \( K > 1 \), and for each \( j \), let \( g_j \) denote a Möbius self-mapping of \( H \) for which \( g_j(a_j) = 0 \) and \( g_j(b_j) = \infty \). Next set
\[
h(re^{i\theta}) = \begin{cases} 
    re^{i\theta/K} & \text{if } 0 \leq \theta < \pi, \\
    re^{i\theta} & \text{if } -\pi < \theta \leq 0,
\end{cases}
\]
and
\[
g(z) = \begin{cases} 
    g_j^{-1} \circ h \circ g_j(z) & \text{if } z \in B_j, \\
    z & \text{if } z \in L.
\end{cases}
\]
Then \( g \) is a homeomorphism of \( D \) onto a quasidisk. We must show \( g|B \) has a \( K_0 \)-quasiconformal extension \( g_0 \) to \( \overline{C} \) for each open disk \( B \) in \( D \) where \( K_0(K) \to 1 \) as \( K \to 1 \). If \( B \subset L \), we may take \( g_0(z) = z \). If \( B \cap B_j \cap H \neq \emptyset \), then \( g_j(B) \) lies in some half-plane
\[
G = \left\{ re^{i\theta} : |\theta - \phi| < \frac{\pi}{2}, \ 0 < r < \infty \right\}, \quad |\phi| < \frac{\pi}{2},
\]
and \( h|G \) can be extended by means of a simple folding to a \( K \)-quasiconformal mapping \( h_0 \) of \( \overline{C} \), and \( g_0 = g_j^{-1} \circ h_0 \circ g_j \) is the desired extension of \( g|B \).

Suppose next that \( S_f \) is in \( \pi(T_1) \). Then \( S_f = \pi(\phi''/\phi') \) where \( \phi''/\phi' \) is in \( T_1 \) and there exists a conformal mapping \( \psi \) of \( U \) onto a quasidisk \( D'' \subset C \) with
\[
\sup_{z \in B'} \left| \frac{h''}{h'}(z) \right| \rho_{D'}(z)^{-1} = \left| \frac{h''}{h'} \right|_{D'}^{-1} = \left| \frac{\phi''}{\phi'} - \frac{\psi''}{\psi'} \right|_{U}^{-1} \leq c < 1,
\]
where \( D' = \phi(U) \) and \( h = \psi \circ \phi^{-1} \). Hence
\[
(5.9) \quad \left| \frac{h''}{h'}(z) \right| \leq c \text{ dist}(z, \partial D')^{-1}
\]
for \( z \) in \( D' \). Since \( S_f = S_\phi \), there exists a Möbius transformation \( g \) such that \( \phi = g \circ f \) and hence \( D' = g(D) \). Then, since \( \phi \) has no poles in \( U \), \( g(D) \subset C \), \( g(L) \) is an open disk or half-plane and \( g(B_j) \) is an open disk for \( j \geq j_0 \).

Now fix \( j \geq j_0 \). Then \( \partial g(B_j) \) and \( \partial g(L) \) form acute exterior angles and
\[
(5.10) \quad \frac{\text{dia}(g(C_j))}{|g(a_j) - g(b_j)|} \geq \frac{\text{dia}(C_j)}{|a_j - b_j|},
\]
where \( a_j \) and \( b_j \) are the endpoints of \( C_j \). Thus we obtain
\[
(5.11) \quad \frac{\text{dia}(h \circ g(C_j))}{|h \circ g(a_j) - h \circ g(b_j)|} \geq \frac{1 - c}{16} \left( \frac{\text{dia}(C_j)}{|a_j - b_j|} \right)^{1-c} - 1
\]
from (5.9), (5.10), and Lemma 5.3, with $B = g(B_j)$ and $B' = g(L)$. Then (5.8) and (5.11) imply that $\partial D'' = h \circ g(\partial D)$ is not a quasicircle, and we have a contradiction. □

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Department of Mathematics, University of Helsinki, Helsinki, Finland

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109