PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 110, Number 3, November 1990

A GENERAL MULTIPLIER THEOREM

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(Communicated by J. Marshall Ash)

ABSTRACT. We prove a "multiplier" result for functions of the infinitesimal generator $\mathscr{L}$ of a symmetric semigroup, which generalizes some previous results by E. M. Stein and M. G. Cowling. As an application, we develop a functional calculus for $\mathscr{L}$ in the case when the L^p-operator norm of $\mathscr{L}^{iu}$ has polynomial growth at infinity. In particular, we prove a "multiplier" result of Marcinkiewicz type for functions of $\mathscr{L}$.

Let $\mathscr{L}$ be a positive, possibly unbounded operator (but with dense domain) on $L^2(M)$, where M is a measure space. Let $\{P_\lambda\}$ be the spectral resolution of the identity for which

$$\mathscr{L} f = \int_0^\infty \lambda \, dP_\lambda f, \qquad f \in \mathrm{Dom}(\mathscr{L}).$$

For every positive real number t, we define the operator T_t by the formula

$$T_t f = \int_0^\infty e^{-t\lambda} \, dP_\lambda f, \qquad f \in L^2(M).$$

We assume that T_t satisfies the contraction property

$$\|T_t f\|_p \le \|f\|_p, \quad f \in L^2(M) \cap L^p(M), \tag{1}$$

whenever $1 \le p \le \infty$. We say that $(T_t)_{t>0}$ is a symmetric contraction semigroup if it possesses the above properties.

Let m be a complex-valued, bounded, Borel-measurable function on $R_+ \cup \{0\}$. We define the "multiplier operator" $m(\mathscr{L})$ by the rule

$$m(\mathscr{L}) f = \int_0^\infty m(\lambda) \, dP_\lambda f, \qquad f \in L^2(M). \tag{2}$$

By spectral theory, $m(\mathscr{L})$ is a bounded operator on $L^2(M)$. Our concern is to find conditions on the function m which ensure that the operator $m(\mathscr{L})$, initially well-defined on $L^2(M) \cap L^p(M)$, extends to a bounded operator on $L^p(M)$ for some p, $1 < p < \infty$.

Received by the editors April 28, 1989 and, in revised form, October 2, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42B15, 47A60; Secondary 47D05.
Key words and phrases. Multiplier operator, Mellin transform, g-function.

The first general result in this direction was obtained by E. M. Stein [13, Corollary 3, p. 121]. He proved, under some additional hypotheses on the semigroup $(T_t)_{t>0}$, that if $1 < p < \infty$ and m is of Laplace transform type, then $m(\mathscr{L})$ extends to a bounded operator on $L^p(M)$.

M. G. Cowling [2] improved Stein's theorem by showing, under our hypotheses on the semigroup $(T_t)_{t>0}$, that if m extends to a bounded holomorphic function on the cone Γ_ψ, where

$$\Gamma_\psi = \{z \in C\colon |\arg z| < \psi\},$$

then $m(\mathscr{L})$ is bounded on $L^p(M)$, provided that $|1/p - 1/2| < \psi/\pi$.

Our result involves a more quantitative condition and can be stated as follows. Given a positive integer N, we denote by $m_N(t, \lambda)$ the function defined by the rule

$$m_N(t, \lambda) = (t\lambda)^N e^{-t\lambda/2} m(\lambda), \qquad t, \lambda > 0 \tag{3}$$

and by $\mathscr{M}_N(t, u)$ the Mellin transform of $m_N(t, \lambda)$ with respect to the variable λ; i.e.,

$$\mathscr{M}_N(t, u) = \int_0^\infty \frac{d\lambda}{\lambda} \lambda^{-iu} m_N(t, \lambda).$$

Theorem 1. *Assume that for some p, $1 < p < \infty$, and for some positive integer N, the following condition holds*:

$$\int_{\mathbf{R}} du \sup_{t>0} |\mathscr{M}_N(t, u)| \; ||| \mathscr{L}^{iu} |||_p < \infty, \tag{4}$$

where $||| \mathscr{L}^{iu} |||_p$ denotes the norm of $\mathscr{L}^{iu}$ as a operator on $L^p(M)$. Then $m(\mathscr{L})$ extends to a bounded operator on $L^p(M)$.

Notice that condition (4) makes sense, since the operator $\mathscr{L}^{iu}$, $u \in R$, defined by the rule

$$\mathscr{L}^{iu} f = \int_0^\infty \lambda^{iu}\, dP_\lambda f, \qquad f \in L^2(M),$$

is, in fact, bounded on $L^p(M)$, $1 < p < \infty$ [13, Corollary 4, p. 121]. Moreover Cowling proved that the operator norm of $\mathscr{L}^{iu}$ satisfies the following estimate.

$$||| \mathscr{L}^{iu} |||_p \le C(p)(1 + |u|^3 \log|u|)^{|1/p - 1/2|} \exp(\pi |1/p - 1/2|\, |u|) \tag{5}$$

[2]. Our result gives an answer to a question posed by Stein in [13].

We do not get H^1 results. H^1 results have been obtained by N. T. Varopoulos by probabilistic methods, under stronger hypotheses than ours [14].

Our result is proved via a g-function argument and the "Mellin transform technique." The proof is in §1. As a consequence, we give a different proof of Cowling's result. In §2 we restrict ourselves to a rather special situation. We assume that $||| \mathscr{L}^{iu} |||_p$ has only polynomial growth at infinity. This condition is fulfilled in many concrete cases such as stratified groups (here $\mathscr{L}$ is essentially the subLaplacean) and compact manifolds (here $\mathscr{L}$ is any elliptic positive self-adjoint operator). We show that if m satisfies a suitable "Hörmander condition" on R_+, then $m(\mathscr{L})$ is bounded on $L^p(M)$, $1 < p < \infty$. Results of this type on stratified groups are well known [3, 4, 6, 8, 9]. Their proofs, however, depend essentially on the dilation structure of the group. Our contribution is to prove that an L^p-functional calculus can be developed, without appealing to any geometric property of the underlying space M.

1. The general case

Throughout the paper we assume that the spectral projection P_0 is trivial on $L^p(M)$, $1 \leq p < \infty$. Then we do not worry about the definition of $m(0)$. We recall some facts that will be needed in the sequel.

Let N be a positive integer. Then, given a function f in $L^p(M)$, we define $g_N(f)$ by the formula

$$g_N(f) = \left(\int_0^\infty \frac{dt}{t}\left|t^N \frac{d^N}{dt^N} T_t f\right|^2\right)^{1/2}. \tag{6}$$

The L^p theory of the Littlewood–Paley function in several variables was developed by Stein in [11], while the L^p theory of the g_N-function for semigroups was developed by Stein in [13, Corollaries 1 and 2, p. 120] and then refined by R. R. Coifman, R. Rochberg, and G. Weiss [1]. We state the following result for easy reference.

Theorem 2. *Suppose that $1 < p < \infty$ and that N is a positive integer. Then there exist positive constants $A_{p,N}$, $B_{p,N}$ such that*

$$A_{p,N}\|f\|_p \leq \|g_N(f)\|_p \leq B_{p,N}\|f\|_p, \qquad f \in L^p(M).$$

Proof. The proof is an immediate consequence of the fact that the operators T_t are subpositive [5, Lemma VIII.6.4] and of the results by Stein and Coifman, Rochberg, and Weiss [1].

We are now in position to prove our main result.

Proof of Theorem 1. The idea of the proof is taken from Stein's proof of the Mikhlin–Hörmander multiplier theorem [12, p. 96]. Take a function f in $\mathrm{Dom}(\mathscr{L}^{N+1}) \cap L^p(M)$. By spectral theory and the semigroup property, we have that

$$t^{N+1}\frac{d^{N+1}}{dt^{N+1}} T_t(m(\mathscr{L})f) = t\mathscr{L}T_{t/2}(T_{t/2}(t\mathscr{L})^N m(\mathscr{L})f).$$

Again by spectral theory,

$$T_{t/2}(t\mathscr{L})^N m(\mathscr{L})f = \int_0^\infty m_N(t, \lambda)\, dP_\lambda f,$$

whence

$$T_{t/2}(t\mathscr{L})^N m(\mathscr{L})f = (2\pi)^{-1}\int_{\mathbf{R}} du\, \mathscr{M}_N(t, u)\mathscr{L}^{iu} f.$$

It is also clear that

$$T_{t/2}(T_{t/2}(t\mathscr{L})^N m(\mathscr{L})f) = (2\pi)^{-1}\int_{\mathbf{R}} du\,\mathscr{M}_N(t, u)T_{t/2}\mathscr{L}^{iu}f.$$

Since $\mathscr{L}$ is a closed operator, $T_{t/2}(\mathscr{L}^{iu}f)$ is in $\operatorname{Dom}(\mathscr{L})$ and

$$\int_{\mathbf{R}} du|\mathscr{M}_N(t, u)|\,\|\mathscr{L}T_{t/2}(\mathscr{L}^{iu}f)\|_2 < \infty,$$

by (4), we have that

$$\begin{aligned} t\mathscr{L}T_{t/2}(T_{t/2}(t\mathscr{L})^N m(\mathscr{L})f) &= (2\pi)^{-1}\int_{\mathbf{R}} du\,\mathscr{M}_N(t, u)t\mathscr{L}T_{t/2}(\mathscr{L}^{iu}f) \\ &= (2\pi)^{-1}\int_{\mathbf{R}} du\,\mathscr{M}_N(t, u)t\frac{d}{dt}T_{t/2}(\mathscr{L}^{iu}f). \end{aligned}$$

Set $C_{p,N} = (2\pi A_{p,N+1})^{-1}$. Then, at least formally, we have that

$$\begin{aligned} \|m(\mathscr{L})f\|_p &\le A_{p,N+1}^{-1}\|g_{N+1}(m(\mathscr{L})f)\|_p \\ &= C_{p,N}\left\|\left(\int_0^\infty \frac{dt}{t}\left|\int_{\mathbf{R}} du\,\mathscr{M}_N(t, u)t\frac{d}{dt}T_{t/2}(\mathscr{L}^{iu}f)\right|^2\right)^{1/2}\right\|_p \\ &\le C_{p,N}\int_{\mathbf{R}} du\sup_{t>0}|\mathscr{M}_N(t, u)|\,\|g_1(\mathscr{L}^{iu}f)\|_p \\ &\le B_{p,1}C_{p,N}\left(\int_{\mathbf{R}} du\sup_{t>0}|\mathscr{M}_N(t, u)|\;|||\,\mathscr{L}^{iu}\,|||_p\right)\|f\|_p. \end{aligned}$$

It is not hard to justify the formal steps above. By (4), the last expression converges. Therefore

$$\int_{\mathbf{R}} du\,\mathscr{M}_N(t, u)t\frac{d}{dt}T_{t/2}(\mathscr{L}^{iu}f)$$

converges as a $L^p(M, L^2((0, \infty), \frac{dt}{t}))$-valued integral as long as the map

$$u \mapsto \mathscr{M}_N(t, u)t\frac{d}{dt}T_{t/2}(\mathscr{L}^{iu}f)$$

is measurable. In fact, we can show that it is continuous. The continuity is a simple consequence of the continuity of the map $u \mapsto \mathscr{L}^{iu}$ in the strong operator topology on $L^p(M)$, the trivial inequality

$$|\mathscr{M}_N(t, u)| \le C_N\|m\|_\infty,$$

the L^p-theory of the g-function and the Lebesgue-dominated convergence theorem.

Now, a density argument completes the proof.

We now look for easily verifiable conditions on m which ensure that (4) holds. It is clear that in the general case (i.e., when no restriction is made on the growth of $|||\,\mathscr{L}^{iu}\,|||_p$), m should be analytic in a neighborhood of the spectrum of $\mathscr{L}$. As an application of Theorem 1, we give a different proof of Cowling's result.

Theorem 3. *Assume that* m *extends to a bounded analytic function in the cone*

$$\Gamma_\psi = \{z \in C : |\arg z| < \psi\}.$$

Then $m(\mathscr{L})$ *extends to a bounded operator on* $L^p(M)$, *provided that* $|1/p - 1/2| < \psi/\pi$.

Proof. We first prove the result at the endpoint $\psi = \pi/2$. The general case follows from this by interpolation, as in [2]. Let N be an integer ≥ 2. Let M be the bounded function defined by the rule

$$M(y) = \lim_{x \to 0} m(x + iy).$$

Then

$$m(x+iy) = \frac{1}{\pi} \int_{\mathbf{R}} ds \frac{x}{x^2 + (y-s)^2} M(s), \qquad x > 0,$$

whence

$$m(x) = \frac{1}{\pi} \int_{\mathbf{R}} ds \frac{x}{x^2 + s^2} M(s), \qquad x > 0. \tag{7}$$

Changing variables and taking (7) into account, we have that

$$\begin{aligned}
\mathscr{M}_N(t, u) &= t^{iu} \int_0^\infty dv\, v^{N-iu-1} e^{-v/2} m\left(\frac{v}{t}\right) \\
&= \frac{t^{iu}}{\pi} \int_0^\infty dv\, v^{N-iu-1} e^{-v/2} \frac{v}{t} \int_{\mathbf{R}} ds \frac{M(s)}{(v/t)^2 + s^2} \\
&= \frac{t^{iu-1}}{\pi} \int_{\mathbf{R}} ds\, M(s) \int_0^\infty dv \frac{v^{N-iu} e^{-v/2}}{(v/t)^2 + s^2} \\
&= \frac{t^{1+iu}}{\pi} \int_{\mathbf{R}} ds\, M(s) \int_0^\infty dv \frac{v^{N-iu} e^{-v/2}}{v^2 + (ts)^2} \\
&= \frac{t^{iu}}{\pi} \int_{\mathbf{R}} d\omega\, M\left(\frac{\omega}{t}\right) \int_0^\infty dv \frac{v^{N-iu} e^{-v/2}}{v^2 + \omega^2}.
\end{aligned}$$

Since

$$\frac{1}{\pi} \frac{v}{v^2 + \omega^2} = \int_{\mathbf{R}} dx\, e^{-v|x| - i\omega x}$$

we get that

$$\begin{aligned}
\mathscr{M}_N(t, u) &= t^{iu} \int_{\mathbf{R}} d\omega\, M\left(\frac{\omega}{t}\right) \int_0^\infty dv\, v^{N-iu-1} e^{-v/2} \int_{\mathbf{R}} dx\, e^{-v|x| - i\omega x} \\
&= t^{iu} \int_{\mathbf{R}} d\omega\, M\left(\frac{w}{t}\right) \int_{\mathbf{R}} dx\, e^{-i\omega x} \int_0^\infty dv\, v^{N-iu-1} e^{-v(1/2 + |x|)} \\
&= t^{iu} \Gamma(N - iu) \int_{\mathbf{R}} d\omega\, M\left(\frac{\omega}{t}\right) \int_{\mathbf{R}} dx \left(\frac{1}{2} + |x|\right)^{-N+iu} e^{-i\omega x}.
\end{aligned}$$

Set $\phi_{N,u}(x) = (\frac{1}{2} + |x|)^{iu-N}$; its distributional derivative is the (L^2) function $\phi'_{N,u}(x) = (iu - N)(\frac{1}{2} + |x|)^{iu-N-1} \operatorname{sgn} x$, where $\operatorname{sgn} x$ is the function which equals -1 if $x < 0$ and 1 if $x > 0$. By the classical Bernstein inequality, we have that

$$\begin{aligned}\|\hat{\phi}_{N,u}\|_{L^1(\mathbf{R})} &\leq C(\|\phi_{N,u}\|^2_{L^2(\mathbf{R})} + \|\phi'_{N,u}\|^2_{L^2(\mathbf{R})})^{1/2} \\ &\leq C_N(1+|u|).\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{\mathbf{R}} d\omega \left|M\left(\frac{\omega}{t}\right)\right| \left|\int_{\mathbf{R}} dx \left(\frac{1}{2}+|x|\right)^{-N+iu} e^{-i\omega x}\right| \\ &\quad \leq C\|\hat{\phi}_{N,u}\|_{L^1(\mathbf{R})}\|M\|_\infty \\ &\quad \leq C_N(1+|u|)\|m\|_{H^\infty(\Gamma_{\pi/2})};\end{aligned}$$

hence, by (5) and the asymptotic behavior of $\Gamma(N-iu)$ when $|u| \to \infty$, we have that

$$\begin{aligned}&\int_{\mathbf{R}} du \sup_{t>0} |\mathscr{M}_N(t,u)| \; ||| \mathscr{L}^{iu} |||_p \\ &\quad \leq C_N \left(\int_{\mathbf{R}} du|\Gamma(N-iu)|(1+|u|)(1+|u|^3\log|u|)^{|1/p-1/2|}\right. \\ &\qquad\qquad \left.\times \exp(\pi|1/p-1/2|\,|u|)\right) \|m\|_{H^\infty(\Gamma_{\pi/2})} \\ &\quad \leq C_N \left(\int_{\mathbf{R}} du \exp(-\delta_p|u|)\right) \|m\|_{H^\infty(\Gamma_{\pi/2})}\end{aligned}$$

for some positive δ_p. Therefore (4) is satisfied and the statement relative to the case $\psi = \pi/2$ is proved.

2. The polynomial growth case

Throughout this section we assume that there exists a positive real number β such that

$$||| \mathscr{L}^{iu} |||_p \leq C(p)(1+|u|)^{\beta|1/p-1/2|}, \qquad u \in R \tag{8}$$

whenever $1 < p < \infty$. We say that m satisfies a Hörmander condition of order α if

(i) m is bounded
(ii) $\sup_{R>0} \int_{R/2}^{2R} \frac{d\lambda}{\lambda} |\lambda^j m^{(j)}(\lambda)|^2 \leq C$ whenever $j = 1, \ldots, \alpha$.

This kind of condition was introduced by Hörmander in [6].

Theorem 4. *Suppose that m satisfies a Hörmander condition of order α, for some $\alpha > \beta/2 + 1$. Then $m(\mathscr{L})$ extends to a bounded operator on $L^p(M)$, $1 < p < \infty$.*

Proof. We show that m satisfies (4). Let N be an integer greater than α. Let ψ be a $C_0^\infty(R)$ function supported in $[1/2, 2]$ and such that

$$\sum_{k=-\infty}^{\infty} \psi(2^k\lambda) = 1, \qquad \lambda \in R_+.$$

Set

$$\gamma(N, \alpha; u) = \frac{(-1)^{\alpha}}{(N-iu)\cdots(N-iu+\alpha-1)}. \tag{9}$$

Integrating by parts α times, we have that

(10)

$$\mathscr{M}_N(t, u) = \gamma(N, \alpha; u)t^{iu}\sum_k \int_0^{\infty} \frac{d\lambda}{\lambda}\lambda^{N+\alpha-iu}\frac{d^{\alpha}}{d\lambda^{\alpha}}\left(e^{-\lambda/2}m\left(\frac{\lambda}{t}\right)\psi(2^k\lambda)\right).$$

By Leibniz's rule, the derivative of order α in the integral can essentially be written as a weighted sum of terms of the form

$$e^{-\lambda/2}\frac{1}{t^s}m^{(s)}\left(\frac{\lambda}{t}\right)2^{kr}\psi^{(r)}(2^k\lambda),$$

where $0 \leq s+r \leq \alpha$; $s, r \geq 0$. Set

$$I_{k,\alpha,N}(t) = \int_0^{\infty}\frac{d\lambda}{\lambda}\lambda^{N+\alpha-iu}e^{-\lambda/2}\frac{1}{t^s}m^{(s)}\left(\frac{\lambda}{t}\right)2^{kr}\psi^{(r)}(2^k\lambda).$$

We claim that

$$\sup_{t>0}|I_{k,\alpha,N}(t)| \leq C\begin{cases} 2^{-k\alpha} & \text{if } k>0 \\ 2^{-k(N+\alpha)}\exp(-2^{-k-1}) & \text{if } k \leq 0. \end{cases} \tag{11}$$

We postpone for a moment the proof of (11). If (11) holds, then

$$\begin{aligned}\sup_{t>0}|\mathscr{M}_N(t, u)| &\leq C(1+|u|)^{-\alpha}\sum_k \sup_{t>0}|I_{k,\alpha,N}(t)| \\ &\leq C(1+|u|)^{-\alpha},\end{aligned}$$

whence

$$\int_{\mathbf{R}} du \sup_{t>0}|\mathscr{M}_N(t, u)| \; ||| \mathscr{L}^{iu} |||_p \leq C\int_{\mathbf{R}} du(1+|u|)^{-\alpha+\beta|1/p-1/2|} \leq C.$$

Therefore (4) holds and the desired result follows from Theorem 1. It remains to prove (11). Changing variables and using Schwartz's inequality, we get that

$$\begin{aligned}|I_{k,\alpha,N}(t)| &= \left|\int_{1/2}^{2}\frac{d\lambda}{\lambda}\left(\frac{\lambda}{2^k}\right)^{N+\alpha-s-iu}\frac{2^{kr}}{e^{\lambda/2^{k+1}}}\left(\frac{\lambda}{2^kt}\right)^s m^{(s)}\left(\frac{\lambda}{2^kt}\right)\psi^{(r)}(\lambda)\right| \\ &\leq C2^{-k(N+\alpha-s-r)}\left(\int_{1/2}^{2}\frac{d\lambda}{\lambda}|\lambda^{N+\alpha-s}e^{-\lambda/2^{k+1}}|^2\right)^{1/2} \\ &\quad\times\left(\int_{1/2}^{2}\left|\left(\frac{\lambda}{2^kt}\right)^s m^{(s)}\left(\frac{\lambda}{2^kt}\right)\right|^2\frac{d\lambda}{\lambda}\right)^{1/2}.\end{aligned}$$

Clearly

$$\left(\int_{1/2}^{2}\frac{d\lambda}{\lambda}\lambda^{2(N+\alpha-s)}e^{-\lambda/2^k}\right)^{1/2} \leq C\begin{cases} 1 & \text{if } k>0 \\ \exp(-2^{-k-1}) & \text{if } k \leq 0. \end{cases}$$

Since m satisfies a Hörmander condition of order α, (11) follows readily from the above estimates. The proof is complete.

Corollary 1. *Suppose that m satisfies the following condition*

$$\sup_{\lambda>0} |\lambda^j m^{(j)}(\lambda)| \leq C, \qquad j = 0, 1, \ldots, \alpha \tag{12}$$

for some $\alpha > \beta/2 + 1$. Then $m(\mathscr{L})$ extends to a bounded operator on $L^p(M)$, $1 < p < \infty$.

Proof. It is obvious that if m satisfies the condition (12), then it satisfies a Hörmander condition of order α. Therefore the result follows immediately from Theorem 4.

Remarks. (a) As Cowling pointed out , Theorem 4 and Corollary 1 are not true if $(T_t)_{t>0}$ are bounded semigroups only for $1 < p < \infty$.

(b) J. Peetre [9, Chapter 10], gives L^p "multiplier results" for the polynomial case under a hypothesis even weaker than (8). His results, however, are a different kind than ours.

(c) In the particular case when M is a stratified group and $\mathscr{L}$ is (minus) the sub-Laplacean, there are many multiplier results. L. De Michele and G. Mauceri [3] give a result for the Heisenberg group. The first result for general stratified groups is due to Hulanicki and Stein [5, Theorem 6.25, p. 208]. They require more regularity assumptions on m and their proof exploits the dilation structure of the group. On the other hand, they also obtain H^p results $(p < 1)$. In this context there are also some relevant results of Hulanicki and Jenkins [7]. The result by Hulanicki and Stein has been subsequently improved by G. Mauceri [8], who weakened the regularity assumption on m and introduced also some local Besov spaces. He obtained H^p results for $p \geq 1$. The case $0 < p < 1$ was carried over by De Michele and Mauceri in [3]. A better multiplier result has recently been obtained by G. Mauceri and the author.

Acknowledgments

It is a pleasure to thank Giancarlo Mauceri for many illuminating conversations on the subject of this paper and for greatly stimulating my work. Thanks are also extended to Michael Cowling for some interesting suggestions and comments.

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