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A GENERAL MULTIPLIER THEOREM

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ABSTRACT. We prove a "multiplier" result for functions of the infinitesimal generator $\mathscr{L}$ of a symmetric semigroup, which generalizes some previous results by E. M. Stein and M. G. Cowling. As an application, we develop a functional calculus for $\mathscr{L}$ in the case when the L^p-operator norm of $\mathscr{L}^{iu}$ has polynomial growth at infinity. In particular, we prove a "multiplier" result of Marcinkiewicz type for functions of $\mathscr{L}$.

Let $\mathscr{L}$ be a positive, possibly unbounded operator (but with dense domain) on $L^2(M)$, where M is a measure space. Let $\{P_\lambda\}$ be the spectral resolution of the identity for which

$$\mathscr{L}f = \int_0^\infty \lambda \, dP_\lambda f, \qquad f \in \mathrm{Dom}(\mathscr{L}).$$

For every positive real number t, we define the operator T_t by the formula

$$T_t f = \int_0^\infty e^{-t\lambda} \, dP_\lambda f, \qquad f \in L^2(M).$$

We assume that T_t satisfies the contraction property

$$\|T_t f\|_p \leq \|f\|_p, \quad f \in L^2(M) \cap L^p(M), \tag{1}$$

whenever $1 \leq p \leq \infty$. We say that $(T_t)_{t>0}$ is a symmetric contraction semigroup if it possesses the above properties.

Let m be a complex-valued, bounded, Borel-measurable function on $R_+ \cup \{0\}$. We define the "multiplier operator" $m(\mathscr{L})$ by the rule

$$m(\mathscr{L})f = \int_0^\infty m(\lambda) \, dP_\lambda f, \qquad f \in L^2(M). \tag{2}$$

By spectral theory, $m(\mathscr{L})$ is a bounded operator on $L^2(M)$. Our concern is to find conditions on the function m which ensure that the operator $m(\mathscr{L})$, initially well-defined on $L^2(M) \cap L^p(M)$, extends to a bounded operator on $L^p(M)$ for some p, $1 < p < \infty$.

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The first general result in this direction was obtained by E. M. Stein [13, Corollary 3, p. 121]. He proved, under some additional hypotheses on the semigroup $(T_t)_{t>0}$, that if $1 < p < \infty$ and m is of Laplace transform type, then $m(\mathscr{L})$ extends to a bounded operator on $L^p(M)$.

M. G. Cowling [2] improved Stein's theorem by showing, under our hypotheses on the semigroup $(T_t)_{t>0}$, that if m extends to a bounded holomorphic function on the cone Γ_ψ, where

$$\Gamma_\psi = \{z \in C : |\arg z| < \psi\},$$

then $m(\mathscr{L})$ is bounded on $L^p(M)$, provided that $|1/p - 1/2| < \psi/\pi$.

Our result involves a more quantitative condition and can be stated as follows. Given a positive integer N, we denote by $m_N(t, \lambda)$ the function defined by the rule

$$m_N(t, \lambda) = (t\lambda)^N e^{-t\lambda/2} m(\lambda), \qquad t, \lambda > 0 \tag{3}$$

and by $\mathscr{M}_N(t, u)$ the Mellin transform of $m_N(t, \lambda)$ with respect to the variable λ; i.e.,

$$\mathscr{M}_N(t, u) = \int_0^\infty \frac{d\lambda}{\lambda} \lambda^{-iu} m_N(t, \lambda).$$

Theorem 1. *Assume that for some* p, $1 < p < \infty$, *and for some positive integer* N, *the following condition holds*:

$$\int_{\mathbf{R}} du \sup_{t>0} |\mathscr{M}_N(t, u)| \, |||\, \mathscr{L}^{iu} \,|||_p < \infty, \tag{4}$$

where $|||\, \mathscr{L}^{iu} \,|||_p$ *denotes the norm of* $\mathscr{L}^{iu}$ *as a operator on* $L^p(M)$. *Then* $m(\mathscr{L})$ *extends to a bounded operator on* $L^p(M)$.

Notice that condition (4) makes sense, since the operator $\mathscr{L}^{iu}$, $u \in R$, defined by the rule

$$\mathscr{L}^{iu} f = \int_0^\infty \lambda^{iu} \, dP_\lambda f, \qquad f \in L^2(M),$$

is, in fact, bounded on $L^p(M)$, $1 < p < \infty$ [13, Corollary 4, p. 121]. Moreover Cowling proved that the operator norm of $\mathscr{L}^{iu}$ satisfies the following estimate.

$$|||\, \mathscr{L}^{iu} \,|||_p \le C(p)(1 + |u|^3 \log|u|)^{|1/p - 1/2|} \exp(\pi|1/p - 1/2| \, |u|) \tag{5}$$

[2]. Our result gives an answer to a question posed by Stein in [13].

We do not get H^1 results. H^1 results have been obtained by N. T. Varopoulos by probabilistic methods, under stronger hypotheses than ours [14].

Our result is proved via a g-function argument and the "Mellin transform technique." The proof is in §1. As a consequence, we give a different proof of Cowling's result. In §2 we restrict ourselves to a rather special situation. We assume that $|||\, \mathscr{L}^{iu} \,|||_p$ has only polynomial growth at infinity. This condition is fulfilled in many concrete cases such as stratified groups (here $\mathscr{L}$ is essentially the subLaplacean) and compact manifolds (here $\mathscr{L}$ is any elliptic positive self-adjoint operator). We show that if m satisfies a suitable "Hörmander condition" on R_+, then $m(\mathscr{L})$ is bounded on $L^p(M)$, $1 < p < \infty$. Results of this type on stratified groups are well known [3, 4, 6, 8, 9]. Their proofs, however, depend essentially on the dilation structure of the group. Our contribution is to prove that an L^p-functional calculus can be developed, without appealing to any geometric property of the underlying space M.

1. The general case

Throughout the paper we assume that the spectral projection P_0 is trivial on $L^p(M)$, $1 \le p < \infty$. Then we do not worry about the definition of $m(0)$. We recall some facts that will be needed in the sequel.

Let N be a positive integer. Then, given a function f in $L^p(M)$, we define $g_N(f)$ by the formula

$$g_N(f) = \left(\int_0^\infty \frac{dt}{t} \left| t^N \frac{d^N}{dt^N} T_t f \right|^2 \right)^{1/2}. \tag{6}$$

The L^p theory of the Littlewood–Paley function in several variables was developed by Stein in [11], while the L^p theory of the g_N-function for semigroups was developed by Stein in [13, Corollaries 1 and 2, p. 120] and then refined by R. R. Coifman, R. Rochberg, and G. Weiss [1]. We state the following result for easy reference.

Theorem 2. *Suppose that $1 < p < \infty$ and that N is a positive integer. Then there exist positive constants $A_{p,N}$, $B_{p,N}$ such that*

$$A_{p,N} \|f\|_p \le \|g_N(f)\|_p \le B_{p,N} \|f\|_p, \qquad f \in L^p(M).$$

Proof. The proof is an immediate consequence of the fact that the operators T_t are subpositive [5, Lemma VIII.6.4] and of the results by Stein and Coifman, Rochberg, and Weiss [1].

We are now in position to prove our main result.

Proof of Theorem 1. The idea of the proof is taken from Stein's proof of the Mikhlin–Hörmander multiplier theorem [12, p. 96]. Take a function f in $\mathrm{Dom}(\mathscr{L}^{N+1}) \cap L^p(M)$. By spectral theory and the semigroup property, we have that

$$t^{N+1} \frac{d^{N+1}}{dt^{N+1}} T_t(m(\mathscr{L})f) = t\mathscr{L} T_{t/2}(T_{t/2}(t\mathscr{L})^N m(\mathscr{L})f).$$

Again by spectral theory,

$$T_{t/2}(t\mathscr{L})^N m(\mathscr{L}) f = \int_0^\infty m_N(t, \lambda)\, dP_\lambda f,$$

whence

$$T_{t/2}(t\mathscr{L})^N m(\mathscr{L}) f = (2\pi)^{-1} \int_{\mathbf{R}} du\, \mathscr{M}_N(t, u) \mathscr{L}^{iu} f.$$

It is also clear that

$$T_{t/2}(T_{t/2}(t\mathscr{L})^N m(\mathscr{L})f) = (2\pi)^{-1} \int_{\mathbf{R}} du\, \mathscr{M}_N(t, u) T_{t/2} \mathscr{L}^{iu} f.$$

Since $\mathscr{L}$ is a closed operator, $T_{t/2}(\mathscr{L}^{iu} f)$ is in $\mathrm{Dom}(\mathscr{L})$ and

$$\int_{\mathbf{R}} du |\mathscr{M}_N(t, u)| \, \|\mathscr{L} T_{t/2}(\mathscr{L}^{iu} f)\|_2 < \infty,$$

by (4), we have that

$$\begin{aligned} t\mathscr{L} T_{t/2}(T_{t/2}(t\mathscr{L})^N m(\mathscr{L})f) &= (2\pi)^{-1} \int_{\mathbf{R}} du\, \mathscr{M}_N(t, u) t\mathscr{L} T_{t/2}(\mathscr{L}^{iu} f) \\ &= (2\pi)^{-1} \int_{\mathbf{R}} du\, \mathscr{M}_N(t, u) t \frac{d}{dt} T_{t/2}(\mathscr{L}^{iu} f). \end{aligned}$$

Set $C_{p,N} = (2\pi A_{p,N+1})^{-1}$. Then, at least formally, we have that

$$\begin{aligned} \|m(\mathscr{L})f\|_p &\le A_{p,N+1}^{-1} \|g_{N+1}(m(\mathscr{L})f)\|_p \\ &= C_{p,N} \left\| \left(\int_0^\infty \frac{dt}{t} \left| \int_{\mathbf{R}} du\, \mathscr{M}_N(t, u) t \frac{d}{dt} T_{t/2}(\mathscr{L}^{iu} f) \right|^2 \right)^{1/2} \right\|_p \\ &\le C_{p,N} \int_{\mathbf{R}} du \sup_{t>0} |\mathscr{M}_N(t, u)| \, \|g_1(\mathscr{L}^{iu} f)\|_p \\ &\le B_{p,1} C_{p,N} \left(\int_{\mathbf{R}} du \sup_{t>0} |\mathscr{M}_N(t, u)| \; |||\, \mathscr{L}^{iu} \,|||_p \right) \|f\|_p. \end{aligned}$$

It is not hard to justify the formal steps above. By (4), the last expression converges. Therefore

$$\int_{\mathbf{R}} du\, \mathscr{M}_N(t, u) t \frac{d}{dt} T_{t/2}(\mathscr{L}^{iu} f)$$

converges as a $L^p(M, L^2((0, \infty), \frac{dt}{t}))$-valued integral as long as the map

$$u \mapsto \mathscr{M}_N(t, u) t \frac{d}{dt} T_{t/2}(\mathscr{L}^{iu} f)$$

is measurable. In fact, we can show that it is continuous. The continuity is a simple consequence of the continuity of the map $u \mapsto \mathscr{L}^{iu}$ in the strong operator topology on $L^p(M)$, the trivial inequality

$$|\mathscr{M}_N(t, u)| \le C_N \|m\|_\infty,$$

the L^p-theory of the g-function and the Lebesgue-dominated convergence theorem.

Now, a density argument completes the proof.

We now look for easily verifiable conditions on m which ensure that (4) holds. It is clear that in the general case (i.e., when no restriction is made on the growth of $|||\, \mathscr{L}^{iu} \,|||_p$), m should be analytic in a neighborhood of the spectrum of $\mathscr{L}$. As an application of Theorem 1, we give a different proof of Cowling's result.

Theorem 3. *Assume that* m *extends to a bounded analytic function in the cone*

$$\Gamma_\psi = \{z \in C : |\arg z| < \psi\}.$$

Then $m(\mathscr{L})$ *extends to a bounded operator on* $L^p(M)$, *provided that* $|1/p - 1/2| < \psi/\pi$.

Proof. We first prove the result at the endpoint $\psi = \pi/2$. The general case follows from this by interpolation, as in [2]. Let N be an integer ≥ 2. Let M be the bounded function defined by the rule

$$M(y) = \lim_{x \to 0} m(x + iy).$$

Then

$$m(x+iy) = \frac{1}{\pi}\int_{\mathbf{R}} ds \frac{x}{x^2 + (y-s)^2} M(s), \qquad x > 0,$$

whence

$$m(x) = \frac{1}{\pi}\int_{\mathbf{R}} ds \frac{x}{x^2 + s^2} M(s), \qquad x > 0. \tag{7}$$

Changing variables and taking (7) into account, we have that

$$\begin{aligned}
\mathscr{M}_N(t, u) &= t^{iu} \int_0^\infty dv\, v^{N-iu-1} e^{-v/2} m\left(\frac{v}{t}\right) \\
&= \frac{t^{iu}}{\pi} \int_0^\infty dv\, v^{N-iu-1} e^{-v/2} \frac{v}{t} \int_{\mathbf{R}} ds \frac{M(s)}{(v/t)^2 + s^2} \\
&= \frac{t^{iu-1}}{\pi} \int_{\mathbf{R}} ds\, M(s) \int_0^\infty dv \frac{v^{N-iu} e^{-v/2}}{(v/t)^2 + s^2} \\
&= \frac{t^{1+iu}}{\pi} \int_{\mathbf{R}} ds\, M(s) \int_0^\infty dv \frac{v^{N-iu} e^{-v/2}}{v^2 + (ts)^2} \\
&= \frac{t^{iu}}{\pi} \int_{\mathbf{R}} d\omega\, M\left(\frac{\omega}{t}\right) \int_0^\infty dv \frac{v^{N-iu} e^{-v/2}}{v^2 + \omega^2}.
\end{aligned}$$

Since

$$\frac{1}{\pi}\frac{v}{v^2 + \omega^2} = \int_{\mathbf{R}} dx\, e^{-v|x| - i\omega x}$$

we get that

$$\begin{aligned}
\mathscr{M}_N(t, u) &= t^{iu} \int_{\mathbf{R}} d\omega\, M\left(\frac{\omega}{t}\right) \int_0^\infty dv\, v^{N-iu-1} e^{-v/2} \int_{\mathbf{R}} dx\, e^{-v|x| - i\omega x} \\
&= t^{iu} \int_{\mathbf{R}} d\omega\, M\left(\frac{w}{t}\right) \int_{\mathbf{R}} dx\, e^{-i\omega x} \int_0^\infty dv\, v^{N-iu-1} e^{-v(1/2+|x|)} \\
&= t^{iu}\Gamma(N - iu) \int_{\mathbf{R}} d\omega\, M\left(\frac{\omega}{t}\right) \int_{\mathbf{R}} dx \left(\frac{1}{2} + |x|\right)^{-N+iu} e^{-i\omega x}.
\end{aligned}$$

Set $\phi_{N,u}(x) = (\frac{1}{2} + |x|)^{iu-N}$; its distributional derivative is the (L^2) function $\phi'_{N,u}(x) = (iu - N)(\frac{1}{2} + |x|)^{iu-N-1} \operatorname{sgn} x$, where $\operatorname{sgn} x$ is the function which equals -1 if $x < 0$ and 1 if $x > 0$. By the classical Bernstein inequality, we have that

$$\begin{aligned}\|\hat{\phi}_{N,u}\|_{L^1(\mathbf{R})} &\le C(\|\phi_{N,u}\|^2_{L^2(\mathbf{R})} + \|\phi'_{N,u}\|^2_{L^2(\mathbf{R})})^{1/2} \\ &\le C_N(1+|u|).\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{\mathbf{R}} d\omega \left|M\left(\frac{\omega}{t}\right)\right| \left|\int_{\mathbf{R}} dx \left(\frac{1}{2} + |x|\right)^{-N+iu} e^{-i\omega x}\right| \\ &\quad \le C\|\hat{\phi}_{N,u}\|_{L^1(\mathbf{R})}\|M\|_\infty \\ &\quad \le C_N(1+|u|)\|m\|_{H^\infty(\Gamma_{\pi/2})};\end{aligned}$$

hence, by (5) and the asymptotic behavior of $\Gamma(N-iu)$ when $|u| \to \infty$, we have that

$$\begin{aligned}&\int_{\mathbf{R}} du \sup_{t>0} |\mathscr{M}_N(t,u)| \; |||\mathscr{L}^{iu}|||_p \\ &\quad \le C_N \left(\int_{\mathbf{R}} du |\Gamma(N-iu)|(1+|u|)(1+|u|^3 \log|u|)^{|1/p-1/2|} \right. \\ &\qquad\qquad \left. \times \exp(\pi|1/p-1/2|\,|u|)\right) \|m\|_{H^\infty(\Gamma_{\pi/2})} \\ &\quad \le C_N \left(\int_{\mathbf{R}} du \exp(-\delta_p|u|)\right) \|m\|_{H^\infty(\Gamma_{\pi/2})}\end{aligned}$$

for some positive δ_p. Therefore (4) is satisfied and the statement relative to the case $\psi = \pi/2$ is proved.

2. The polynomial growth case

Throughout this section we assume that there exists a positive real number β such that

$$|||\mathscr{L}^{iu}|||_p \le C(p)(1+|u|)^{\beta|1/p-1/2|}, \qquad u \in R \tag{8}$$

whenever $1 < p < \infty$. We say that m satisfies a Hörmander condition of order α if

(i) m is bounded

(ii) $\sup_{R>0} \int_{R/2}^{2R} \frac{d\lambda}{\lambda} |\lambda^j m^{(j)}(\lambda)|^2 \le C$ whenever $j = 1, \ldots, \alpha$.

This kind of condition was introduced by Hörmander in [6].

Theorem 4. *Suppose that m satisfies a Hörmander condition of order α, for some $\alpha > \beta/2 + 1$. Then $m(\mathscr{L})$ extends to a bounded operator on $L^p(M)$, $1 < p < \infty$.*

Proof. We show that m satisfies (4). Let N be an integer greater than α. Let ψ be a $C_0^\infty(R)$ function supported in $[1/2, 2]$ and such that

$$\sum_{k=-\infty}^{\infty} \psi(2^k\lambda) = 1, \qquad \lambda \in R_+.$$

Set

$$\gamma(N, \alpha; u) = \frac{(-1)^\alpha}{(N - iu)\cdots(N - iu + \alpha - 1)}. \tag{9}$$

Integrating by parts α times, we have that
(10)

$$\mathscr{M}_N(t, u) = \gamma(N, \alpha; u)t^{iu} \sum_k \int_0^\infty \frac{d\lambda}{\lambda}\lambda^{N+\alpha-iu} \frac{d^\alpha}{d\lambda^\alpha}\left(e^{-\lambda/2} m\left(\frac{\lambda}{t}\right)\psi(2^k\lambda)\right).$$

By Leibniz's rule, the derivative of order α in the integral can essentially be written as a weighted sum of terms of the form

$$e^{-\lambda/2}\frac{1}{t^s}m^{(s)}\left(\frac{\lambda}{t}\right)2^{kr}\psi^{(r)}(2^k\lambda),$$

where $0 \le s + r \le \alpha$; $s, r \ge 0$. Set

$$I_{k,\alpha,N}(t) = \int_0^\infty \frac{d\lambda}{\lambda}\lambda^{N+\alpha-iu}e^{-\lambda/2}\frac{1}{t^s}m^{(s)}\left(\frac{\lambda}{t}\right)2^{kr}\psi^{(r)}(2^k\lambda).$$

We claim that

$$\sup_{t>0}|I_{k,\alpha,N}(t)| \le C\begin{cases} 2^{-k\alpha} & \text{if } k > 0 \\ 2^{-k(N+\alpha)}\exp(-2^{-k-1}) & \text{if } k \le 0. \end{cases} \tag{11}$$

We postpone for a moment the proof of (11). If (11) holds, then

$$\begin{aligned}\sup_{t>0}|\mathscr{M}_N(t, u)| &\le C(1 + |u|)^{-\alpha}\sum_k \sup_{t>0}|I_{k,\alpha,N}(t)| \\ &\le C(1 + |u|)^{-\alpha},\end{aligned}$$

whence

$$\int_{\mathbf{R}} du \sup_{t>0}|\mathscr{M}_N(t, u)|\ |||\mathscr{L}^{iu}|||_p \le C\int_{\mathbf{R}} du(1 + |u|)^{-\alpha+\beta|1/p-1/2|} \le C.$$

Therefore (4) holds and the desired result follows from Theorem 1. It remains to prove (11). Changing variables and using Schwartz's inequality, we get that

$$\begin{aligned}|I_{k,\alpha,N}(t)| &= \left|\int_{1/2}^2 \frac{d\lambda}{\lambda}\left(\frac{\lambda}{2^k}\right)^{N+\alpha-s-iu}\frac{2^{kr}}{e^{\lambda/2^{k+1}}}\left(\frac{\lambda}{2^k t}\right)^s m^{(s)}\left(\frac{\lambda}{2^k t}\right)\psi^{(r)}(\lambda)\right| \\ &\le C2^{-k(N+\alpha-s-r)}\left(\int_{1/2}^2 \frac{d\lambda}{\lambda}|\lambda^{N+\alpha-s}e^{-\lambda/2^{k+1}}|^2\right)^{1/2} \\ &\quad\times\left(\int_{1/2}^2\left|\left(\frac{\lambda}{2^k t}\right)^s m^{(s)}\left(\frac{\lambda}{2^k t}\right)\right|^2\frac{d\lambda}{\lambda}\right)^{1/2}.\end{aligned}$$

Clearly

$$\left(\int_{1/2}^2 \frac{d\lambda}{\lambda}\lambda^{2(N+\alpha-s)}e^{-\lambda/2^k}\right)^{1/2} \le C\begin{cases} 1 & \text{if } k > 0 \\ \exp(-2^{-k-1}) & \text{if } k \le 0. \end{cases}$$

Since m satisfies a Hörmander condition of order α, (11) follows readily from the above estimates. The proof is complete.

Corollary 1. *Suppose that* m *satisfies the following condition*

$$\sup_{\lambda>0}|\lambda^j m^{(j)}(\lambda)| \leq C, \qquad j=0, 1, \ldots, \alpha \tag{12}$$

for some $\alpha > \beta/2+1$. *Then* $m(\mathscr{L})$ *extends to a bounded operator on* $L^p(M)$, $1<p<\infty$.

Proof. It is obvious that if m satisfies the condition (12), then it satisfies a Hörmander condition of order α. Therefore the result follows immediately from Theorem 4.

Remarks. (a) As Cowling pointed out , Theorem 4 and Corollary 1 are not true if $(T_t)_{t>0}$ are bounded semigroups only for $1<p<\infty$.

(b) J. Peetre [9, Chapter 10], gives L^p "multiplier results" for the polynomial case under a hypothesis even weaker than (8). His results, however, are a different kind than ours.

(c) In the particular case when M is a stratified group and $\mathscr{L}$ is (minus) the sub-Laplacean, there are many multiplier results. L. De Michele and G. Mauceri [3] give a result for the Heisenberg group. The first result for general stratified groups is due to Hulanicki and Stein [5, Theorem 6.25, p. 208]. They require more regularity assumptions on m and their proof exploits the dilation structure of the group. On the other hand, they also obtain H^p results $(p<1)$. In this context there are also some relevant results of Hulanicki and Jenkins [7]. The result by Hulanicki and Stein has been subsequently improved by G. Mauceri [8], who weakened the regularity assumption on m and introduced also some local Besov spaces. He obtained H^p results for $p \geq 1$. The case $0<p<1$ was carried over by De Michele and Mauceri in [3]. A better multiplier result has recently been obtained by G. Mauceri and the author.

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