

INVERSE SYSTEMS OF ABSTRACT LEBESGUE SPACES

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(Communicated by William J. Davis)

ABSTRACT. We show that inverse limits exist in the category of L^1 spaces and positive linear contractions between them. This result generalizes the well-known classical results for inverse systems of Choquet simplexes and of L -balls, but our proof is simple and more purely geometrical. The result finds physical application in the study of random fields.

Let Z be a Banach lattice with positive cone C and norm $\|\cdot\|$. We say that Z is an *AL-space* (short for Abstract Lebesgue space) iff $\|\cdot\|$ is affine on C . See [7, §8, p. 112] for further information about *AL-spaces*.

Let f be a linear map between two *AL-spaces* Z_2 and Z_1 . We say that f is a *contraction* iff $\|fx\|_1 \leq \|x\|_2$ for all $x \in C_2$. A positive linear contraction $f: Z_2 \rightarrow Z_1$ is called an *AL-morphism*.

Note that for an *AL-morphism* f we have $\|fx\|_1 \leq \|x\|_2$ for all $x \in Z_2$ since

$$\|fx\|_1 = \|\lfloor fx \rfloor\|_1 \leq \|\lfloor x \rfloor\|_1 \leq \|x\|_2 = \|x\|_2;$$

however, an *AL-morphism* need not be a lattice homomorphism, an isometry, or even one-to-one.

Let D be a directed set, and suppose that we have an *AL-space* Z_i associated with each $i \in D$ and an *AL-morphism* $f_{ji}: Z_i \rightarrow Z_j$ associated with each pair $i, j \in D$ where $i > j$.

If this system is consistent in the sense that

$$f_{ji} = f_{jk} f_{ki} \quad \text{for all } i > k > j \text{ in } D,$$

then we call $(Z_i, f_{ji})_D$ an inverse *AL-system*.

Now suppose further that Z is an *AL-space*, and $f_i: Z \rightarrow Z_i$ for $i \in D$ are *AL-morphisms* satisfying

$$f_j = f_{ji} f_i \quad \text{for all } i > j \text{ in } D.$$

Then we call $(Z, f_i)_D$ a *prefix* of $(Z_i, f_{ji})_D$.

Received by the editors December 9, 1989 and, in revised form, January 9, 1990.
1980 *Mathematics Subject Classification* (1985 Revision). Primary 46B30, 46M40.

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0002-9939/90 \$1.00 + \$.25 per page

If every other prefix $(Y, g_i)_D$ of $(Z_i, f_{ji})_D$ factors through $(Z, f_i)_D$ in the sense that there exists an *AL*-morphism $g: Y \rightarrow Z$ with

$$g_i = f_i g \quad \text{for all } i \in D,$$

then we call $(Z, f_i)_D$ the *inverse AL-limit* of $(Z_i, f_{ji})_D$.

Theorem. *For every inverse AL-system $(Z_i, f_{ji})_D$ the inverse AL-limit $(Z, f_i)_D$ exists and is unique.*

Proof. Clearly the inverse *AL*-limit is unique up to *AL*-isomorphism if it exists. We demonstrate existence by constructing the inverse *AL*-limit explicitly.

Let ΠZ stand for $\prod_{i \in D} Z_i$ and let $\pi_i: \Pi Z \rightarrow Z_i$ denote the coordinate projections. Write x_i for $\pi_i x$ where $x \in \Pi Z$.

Define

$$\|x\| = \sup_{i \in D} \|x_i\|_i \quad \text{for } x \in \Pi Z,$$

$$Z = \{x \in \Pi Z : \|x\| < \infty \text{ and } f_{ji} x_i = x_j \text{ for all } i > j \in D\},$$

$$C = \{x \in Z : x_i \in C_i \text{ for all } i \in D\},$$

and let $f_i: Z \rightarrow Z_i$ be the restriction of π_i to Z .

We shall show that Z equipped with C and $\|\cdot\|$ is an *AL*-space. First we note that (Z, C) is a vector lattice with sum and modulus given by

$$(x + y)_j = x_j + y_j, \quad (|x|)_j = \lim_{i > j} f_{ji} |x|_i$$

where $|\cdot|_i$ denotes modulus in Z_i and the limit is in $\|\cdot\|_j$.

To see that the definition of modulus makes sense we use a property of *AL*-spaces, that every monotone increasing norm-bounded net converges in norm to its least upper bound. This is easy to prove directly, or see [7, 8.2, p. 113].

Clearly $f_{ji} |x|_i$ is a monotone increasing sequence in Z_j with norm bounded by $\|x\|$. Hence this sequence is norm convergent in Z_j , and indeed $f_{ji} (|x|)_i = (|x|)_j$ and $\| |x| \| = \|x\| < \infty$. Thus $|x| \in Z$ and is easily seen to be the least upper bound in Z of $\{x, -x\}$.

It is now easy to verify that $\|\cdot\|$ is a lattice norm for Z and is additive on C . To show the norm complete under these conditions it suffices to prove that every monotone increasing norm bounded net $(x^{(n)})$ in Z has a least upper bound x [7, 8.2, p. 113]. But this is easy, set

$$x_j = \lim_n x_j^{(n)} \quad \text{for } j \in D$$

and argue as in the previous paragraph to show that $x \in Z$ and is the required least upper bound.

This establishes that Z is an *AL*-space. The f_i are obviously *AL*-morphisms so $(Z, f_i)_D$ is a prefix of $(Z_i, f_{ji})_D$. For any other prefix $(Y, g_i)_D$ define $g: Y \rightarrow Z$ by $(gy)_i = g_i y$. Verification of the factorization property is now routine. Hence $(Z, f_i)_D$ is the inverse *AL*-limit of $(Z_i, f_{ji})_D$. \square

If the dual of a Banach space can be ordered in such a way as to make it an AL -space, then the unit ball of the dual endowed with the weak- $*$ topology is called an L -ball [5].

If all the AL -spaces Z_i are dual spaces of this form then our theorem implies the result of D. A. Edwards [3, Corollary 5, p. 231] that the inverse limit of a system of L -balls is an L -ball.

If in addition each $S_i = \{x \in C_i : \|x\|_i = 1\}$ is a weak- $*$ closed face of the corresponding L -ball, then the S_i are Choquet simplexes.

Conversely (see for example [2, Corollary 3, p. 411]) every Choquet simplex is affine homeomorphic to a simplex of this form. Each affine continuous map f_{ji} from S_i into S_j then extends naturally to an AL -morphism from Z_i into Z_j (although this AL -morphism may fail to be an isometry.)

Thus our theorem also implies the result of F. Jellett, E. B. Davies, and G. F. Vincent-Smith [4, 1] that the inverse limit of a system of Choquet simplexes is a Choquet simplex.

Results of this type are of interest in statistical mechanics (see for example [8, §4.3]). There, the spaces of measures used to underlie the representations of Gibbs states for Föllmer specifications are AL -spaces, and the corresponding substochastic kernels are AL -morphisms.

Often we are concerned only with geometric properties which are intrinsic to the relevant inverse limit space (see for example [6, §1.6]) and in such cases our theorem establishes sufficient structure to determine the phases.

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