DIVISIBILITY BY 2 OF STIRLING-LIKE NUMBERS

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Abstract. We give a characterization of functions of the form \( f(n) = \nu(n - E) \), where \( \nu(\cdot) \) denotes the exponent of 2, and \( E \) is a 2-adic integer. We show that it applies to the restriction to even or odd integers of the function \( f(n) = \nu(a \cdot 5^n + b \cdot 3^n + c) \), with mild restrictions on \( a, b, \) and \( c \). This function is closely related to divisibility of certain Stirling numbers of the second kind.

Let \( \nu(m) \) denote the exponent of 2 in \( m \), and let \( \mathbb{N} \) denote the set of nonnegative integers. Our first result is a characterization of a certain class of functions. We think of a 2-adic integer as a possibly infinite sum of distinct 2-powers.

**Theorem 1.** Let \( f \) be a function \( \mathbb{N} \to \mathbb{N} \cup \{\infty\} \). Then the following are equivalent:

(i) There is a 2-adic integer \( E \) such that \( f(n) = \nu(n - E) \) for all \( n \);

(ii) For all \( n \) and \( d \),

\[
\begin{align*}
&\begin{cases}
  f(n + 2^d) = d & \text{if } d < f(n), \\
  f(n + 2^d) > f(n) & \text{if } d = f(n), \\
  f(n + 2^d) = f(n) & \text{if } d > f(n);
\end{cases}
\end{align*}
\]

(iii) \( f \) satisfies

(a) for all \( n \), \( f(n + 2^{f(n)}) > f(n) \), and

(b) if \( d < f(n) \) and \( \alpha \) is odd, then \( f(n + \alpha 2^d) = d \).

\( E \) is defined by \( E = \sum_{i \geq 1} 2^{e_i} \) with \( e_1 < e_{i+1} \), where \( e_1 = f(0) \), \( E(k) = \sum_{i=1}^{k} 2^{e_i} \), and \( e_{k+1} = f(E(k)) \). Finally, \( E \) is finite if and only if \( \infty \in f(\mathbb{N}) \).

**Proof.** (i)\( \Rightarrow \) (ii): \( f(n) \) equals the exponent of the smallest 2-power in which \( n \) and \( E \) differ. Then \( n + 2^{f(n)} \) and \( E \) agree in that 2-power as well, while \( n + 2^d \) and \( E \) first differ at \( 2^d \) if \( d < f(n) \) and at \( 2^{f(n)} \) if \( d > f(n) \).

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(ii)⇒(iii): Write $\alpha 2^d = 2^d + \sum 2^{e_i}$, with $e_i > d$. Then $f(n + 2^d) = d$, and adding each $2^{e_i}$ to the argument does not change the value of $f$.

(iii)⇒(i): Define $E$ as in the theorem. Hypothesis (a) applied to $E(i - 1)$ guarantees that $e_i < e_{i+1}$. For any $n$, let $d = \nu(n - E)$, and choose $k$ so that $e_k \leq d < e_{k+1}$. Then $n - E(k) = \alpha 2^d$, with $\alpha$ odd, and so we apply (b) with $n = E(k)$, noting that the hypothesis is satisfied since $f(E(k)) = e_{k+1} > d$. We deduce $f(n) = f(E(k) + \alpha 2^d) = d$. □

Our second result is an application of Theorem 1.

**Theorem 2.** If $a$ is odd, $b \equiv 2 \mod 4$, and $a + b + c \equiv 0 \mod 8$, then there exist 2-adic integers $E$ and $E'$ such that

$$\nu(a \cdot 5^n + b \cdot 3^n + c) = 2 + \begin{cases} \nu(n - E) & \text{if } n \text{ even} \\ \nu(n - E') & \text{if } n \text{ odd.} \end{cases}$$

Let $g(n) = -2 + \nu(a \cdot 5^n + b \cdot 3^n + c)$. If $E = \Sigma 2^{e_i}$ with $e_i < e_{i+1}$ and $E' = \Sigma 2^{e'_i}$ with $e'_i < e'_{i+1}$, then $e_1 = g(0)$, $e_{k+1} = g(E(k))$, $e'_1 = 0$, and $e'_{k+1} = g(E'(k))$.

**Proof.** We use the well-known and easily proved fact that

$$\nu(p^i - 1) = \begin{cases} \nu(4i) & \text{if } p = 5, \text{ or } p = 3 \text{ and } i \text{ even} \\ 1 & \text{if } p = 3 \text{ and } i \text{ odd} \end{cases}$$

to observe that if

$$\nu(a \cdot 5^n + b \cdot 3^n + c) = e + 2,$$

and $\alpha$ is odd and $d > 0$, then

$$\nu(a \cdot 5^{n+\alpha 2^d} + b \cdot 3^{n+\alpha 2^d} + c) = \begin{cases} d + 2 & \text{if } d < e \\ > e + 2 & \text{if } d = e \\ = e + 2 & \text{if } d > e \end{cases}$$

Here we have used the decomposition

$$a \cdot 5^{n+\alpha 2^d} + b \cdot 3^{n+\alpha 2^d} + c = 2^{e+2} \text{ odd} + a5^n(5^{n+2^d} - 1) + b3^n(3^{\alpha 2^d} - 1).$$

Thus $g$ satisfies (a) of Theorem 1(iii) and (b) restricted to $d > 0$. This implies that the functions $f_0$ and $f_1$ defined by

$$f_\varepsilon(n) = g(2n + \varepsilon) - 1, \quad \varepsilon = 0, 1$$

satisfy the hypotheses of Theorem 1(iii). Indeed,

$$f_\varepsilon(n + 2^{\varepsilon(n)}) = g(2n + 2^{\varepsilon(n)+1} + \varepsilon) - 1$$

$$= g(2n + 2^{g(2n+\varepsilon)} + \varepsilon) - 1 > g(2n + \varepsilon) - 1 = f_\varepsilon(n),$$

and if $0 \leq d < f_\varepsilon(n)$, then $0 < d + 1 < g(2n + \varepsilon)$; hence

$$f_\varepsilon(n + \alpha 2^d) = g(2n + \alpha 2^{d+1} + \varepsilon) - 1 = (d + 1) - 1.$$
In order that $f_\varepsilon(m) \geq 0$, we need $g(2m + \varepsilon) > 0$, which follows from

\begin{equation}
\tag{3}
g(2m + \varepsilon) = -2 + \nu(5^\varepsilon a (5^{2m} - 1) + 3^\varepsilon b (3^{2m} - 1) + (5^\varepsilon a + 3^\varepsilon b + c)) , 
\end{equation}

since each of the three terms is divisible by 8, the last since the hypotheses imply $5a + 3b + c \equiv 0 \mod 8$.

Thus by Theorem 1, $f_\varepsilon(n) = \nu(n - E)$ for some 2-adic $E_\varepsilon$, and

\[
g(2n) = \nu(n - E_0) + 1 = \nu(2n - E) \text{ with } E = 2E_0 ,
\]

and

\[
g(2n + 1) = \nu(n - E_1) + 1 = \nu(2n + 1 - E') \text{ with } E' = 2E_1 + 1.
\]

Similar manipulations imply that $E$ and $E'$ satisfy the asserted defining property. \hfill \Box

We remark that if $\nu(a + b + c) = \nu < 3$, then $\nu(a 5^n + b 3^n + c) = \nu$ for all even $n$, and if $\nu(5a + 3b + c) = \nu' < 3$, then $\nu(a 5^n + b 3^n + c) = \nu'$ for all odd $n$. This is immediate from (3). If $a + b$ is even, $a + b + c \equiv 0 \mod 8$, and $5a + 3b + c \equiv 0 \mod 8$, then $\nu(a \cdot 5^n + b \cdot 3^n + c)$ is more complicated.

This work was motivated by a desire to determine the exponent of 2 in some Stirling numbers of the second kind. These exponents are important in algebraic topology [3-5]. We take as our definition

\[
S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (k - i)^n, \quad 1 \leq k \leq n .
\]

See [2] for other formulas and combinatorial descriptions. In particular,

\[
S(n, 5) = \frac{1}{5!} (5^n - 5 \cdot 4^n + 10 \cdot 3^n - 10 \cdot 2^n + 5) .
\]

One would expect the second and fourth terms to be much more highly 2-divisible than the combination of the others, in which case $\nu(S(n, 5))$ would equal $-1 + g(n)$, where

\begin{equation}
\tag{4}
g(n) = -2 + \nu(5^n + 10 \cdot 3^n + 5) .
\end{equation}

Theorem 2 applies to this $g$, and a REDUCE program easily calculates the values of the exponents which are less than 100 to be

\[
e_i : 2, 3, 4, 7, 12, 16, 17, 18, 19, 21, 22, 23, 25, 26, 28, 29, 30, 31, 34, 38, 41, 42, 45, 50, 51, 52, 53, 55, 57, 58, 60, 61, 62, 63, 64, 66, 67, 71, 73, 74, 75, 76, 77, 78, 79, 80, 81, 83, 87, 91, 94, 97, 98, 99 .
\]

\[
e'_i : 0, 1, 2, 3, 4, 8, 11, 14, 16, 19, 20, 25, 27, 28, 29, 30, 35, 37, 39, 40, 44, 47, 48, 50, 53, 54, 57, 58, 60, 61, 62, 66, 68, 69, 70, 71, 73, 76, 78, 79, 83, 85, 89, 91, 94 .
\]

Now, if $E$ is defined from (4) as in Theorem 2, so that the $e_i$'s less than 100 are as above, then

\[
\nu(S(n, 5)) = -1 + \nu(n - E) \text{ for even } n, \text{ provided } \nu(n - E) < n - 1 ,
\]
and similarly for odd \( n \). The first failure of this would be for \( n \) equal to the smallest \( E(k) \) such that \( e_{k+1} \geq E(k) - 1 \). It can be observed that this never happens for \( 5 \leq n < 2^{100} \), and for it to happen subsequently would require the unlikely occurrence of more than \( 2^{94} \) consecutive 0's in the binary expansion of \( E \) or \( E' \).

It is then a simple matter to read off \( \nu(S(n, 5)) \) for \( n \) even and \( n < 2^{100} \) from the smallest 2-power in which \( n \) differs from \( E \), as determined from the list of \( e_i \)'s, and similarly for \( n \) odd. For example, since

\[
1989 = 2^0 + 2^2 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10},
\]

we obtain \( \nu(S(1989, 5)) = -1 + \nu(1989 - E') = 0 \).

A similar discussion to all of this applies to \( S(n, 6) \). If \( k < 5 \), then \( \nu(S(n, k)) = 0 \) or 1 depending on the parity of \( n \), and for \( k > 7 \), \( \nu(S(n, k)) \) is somewhat more complicated to analyze. In work [1] stimulated by an earlier version of this paper, Francis Clarke has generalized this work in a number of directions (larger \( k \), odd primes, and a more general context). I thank him for several useful comments on this work.

References


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