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DIVISIBILITY BY 2 OF STIRLING-LIKE NUMBERS

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ABSTRACT. We give a characterization of functions of the form $f(n) = \nu(n - E)$, where $\nu(-)$ denotes the exponent of 2, and E is a 2-adic integer. We show that it applies to the restriction to even or odd integers of the function $f(n) = \nu(a \cdot 5^n + b \cdot 3^n + c)$, with mild restrictions on a, b, and c. This function is closely related to divisibility of certain Stirling numbers of the second kind.

Let $\nu(m)$ denote the exponent of 2 in m, and let $\mathbf{N}$ denote the set of nonnegative integers. Our first result is a characterization of a certain class of functions. We think of a 2-adic integer as a possibly infinite sum of distinct 2-powers.

Theorem 1. *Let f be a function $\mathbf{N} \to \mathbf{N} \cup \{\infty\}$. Then the following are equivalent:*

(i) *There is a 2-adic integer E such that $f(n) = \nu(n - E)$ for all n;*

(ii) *For all n and d,*

$$f(n + 2^d)\begin{cases} = d & \text{if } d < f(n) \\ > f(n) & \text{if } d = f(n) \\ = f(n) & \text{if } d > f(n); \end{cases}$$

(iii) *f satisfies*

(a) *for all n, $f(n + 2^{f(n)}) > f(n)$, and*

(b) *if $d < f(n)$ and α is odd, then $f(n + \alpha 2^d) = d$.*

E is defined by $E = \sum_{i \geq 1} 2^{e_i}$ with $e_i < e_{i+1}$, where $e_1 = f(0)$, $E(k) = \sum_{i=1}^{k} 2^{e_i}$, and $e_{k+1} = f(E(k))$. Finally, E is finite if and only if $\infty \in f(\mathbf{N})$.

Proof. (i)⇒(ii): $f(n)$ equals the exponent of the smallest 2-power in which n and E differ. Then $n + 2^{f(n)}$ and E agree in that 2-power as well, while $n + 2^d$ and E first differ at 2^d if $d < f(n)$ and at $2^{f(n)}$ if $d > f(n)$.

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(ii)⇒(iii): Write $\alpha 2^d = 2^d + \Sigma 2^{e_i}$, with $e_i > d$. Then $f(n+2^d) = d$, and adding each 2^{e_i} to the argument does not change the value of f.

(iii)⇒(i): Define E as in the theorem . Hypothesis (a) applied to $E(i-1)$ guarantees that $e_i < e_{i+1}$. For any n, let $d = \nu(n-E)$, and choose k so that $e_k \le d < e_{k+1}$. Then $n - E(k) = \alpha 2^d$, with α odd, and so we apply (b) with $n = E(k)$, noting that the hypothesis is satisfied since $f(E(k)) = e_{k+1} > d$. We deduce $f(n) = f(E(k) + \alpha 2^d) = d$. □

Our second result is an application of Theorem 1.

Theorem 2. *If a is odd, $b \equiv 2 \bmod 4$, and $a+b+c \equiv 0 \bmod 8$, then there exist 2-adic integers E and E' such that*

$$\nu(a\cdot 5^n + b\cdot 3^n + c) = 2 + \begin{cases} \nu(n-E) & \text{if } n \text{ even} \\ \nu(n-E') & \text{if } n \text{ odd.} \end{cases}$$

Let $g(n) = -2 + \nu(a\cdot 5^n + b\cdot 3^n + c)$. If $E = \Sigma 2^{e_i}$ with $e_i < e_{i+1}$ and $E' = \Sigma 2^{e'_i}$ with $e'_i < e'_{i+1}$, then $e_1 = g(0)$, $e_{k+1} = g(E(k))$, $e'_1 = 0$, and $e'_{k+1} = g(E'(k))$.

Proof. We use the well-known and easily proved fact that

$$\nu(p^i - 1) = \begin{cases} \nu(4i) & \text{if } p = 5, \text{ or } p = 3 \text{ and } i \text{ even} \\ 1 & \text{if } p = 3 \text{ and } i \text{ odd} \end{cases}$$

to observe that if

$$\nu(a\cdot 5^n + b\cdot 3^n + c) = e + 2,$$

and α is odd and $d > 0$, then

$$\nu(a\cdot 5^{n+\alpha 2^d} + b\cdot 3^{n+\alpha 2^d} + c)\begin{cases} = d+2 & \text{if } d < e \\ > e+2 & \text{if } d = e \\ = e+2 & \text{if } d > e. \end{cases}$$

Here we have used the decomposition

$$a\cdot 5^{n+\alpha 2^d} + b\cdot 3^{n+\alpha 2^d} + c = 2^{e+2}\text{ odd } + a5^n(5^{\alpha 2^d} - 1) + b3^n(3^{\alpha 2^d} - 1).$$

Thus g satisfies (a) of Theorem 1(iii) and (b) restricted to $d > 0$. This implies that the functions f_0 and f_1 defined by

$$f_\varepsilon(n) = g(2n+\varepsilon) - 1, \quad \varepsilon = 0, 1$$

satisfy the hypotheses of Theorem 1(iii). Indeed,

$$\begin{aligned} f_\varepsilon(n + 2^{f_\varepsilon(n)}) &= g(2n + 2^{f_\varepsilon(n)+1} + \varepsilon) - 1 \\ &= g(2n + 2^{g(2n+\varepsilon)} + \varepsilon) - 1 > g(2n+\varepsilon) - 1 = f_\varepsilon(n), \end{aligned}$$

and if $0 \le d < f_\varepsilon(n)$, then $0 < d+1 < g(2n+\varepsilon)$; hence

$$f_\varepsilon(n + \alpha 2^d) = g(2n + \alpha 2^{d+1} + \varepsilon) - 1 = (d+1) - 1.$$

In order that $f_\varepsilon(m) \geq 0$, we need $g(2m+\varepsilon) > 0$, which follows from

$$g(2m+\varepsilon) = -2 + \nu(5^\varepsilon a(5^{2m}-1) + 3^\varepsilon b(3^{2m}-1) + (5^\varepsilon a + 3^\varepsilon b + c)), \tag{3}$$

since each of the three terms is divisible by 8, the last since the hypotheses imply $5a+3b+c \equiv 0 \bmod 8$.

Thus by Theorem 1, $f_\varepsilon(n) = \nu(n - E_\varepsilon)$ for some 2-adic E_ε, and

$$g(2n) = \nu(n - E_0) + 1 = \nu(2n - E) \text{ with } E = 2E_0,$$

and

$$g(2n+1) = \nu(n - E_1) + 1 = \nu(2n+1-E') \text{ with } E' = 2E_1 + 1.$$

Similar manipulations imply that E and E' satisfy the asserted defining property. □

We remark that if $\nu(a+b+c) = v < 3$, then $\nu(a5^n + b3^n + c) = v$ for all even n, and if $\nu(5a+3b+c) = v' < 3$, then $\nu(a5^n + b3^n + c) = v'$ for all odd n. This is immediate from (3). If $a+b$ is even, $a+b+c \equiv 0 \bmod 8$, and $5a+3b+c \equiv 0 \bmod 8$, then $\nu(a \cdot 5^n + b \cdot 3^n + c)$ is more complicated.

This work was motivated by a desire to determine the exponent of 2 in some Stirling numbers of the second kind. These exponents are important in algebraic topology [3–5]. We take as our definition

$$S(n, k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n, \quad 1 \leq k \leq n.$$

See [2] for other formulas and combinatorial descriptions. In particular,

$$S(n, 5) = \frac{1}{5!}(5^n - 5 \cdot 4^n + 10 \cdot 3^n - 10 \cdot 2^n + 5).$$

One would expect the second and fourth terms to be much more highly 2-divisible than the combination of the others, in which case $\nu(S(n, 5))$ would equal $-1 + g(n)$, where

$$g(n) = -2 + \nu(5^n + 10 \cdot 3^n + 5). \tag{4}$$

Theorem 2 applies to this g, and a REDUCE program easily calculates the values of the exponents which are less than 100 to be

e_i : 2, 3, 4, 7, 12, 16, 17, 18, 19, 21, 22, 23, 25, 26, 28, 29, 30, 31, 34, 38, 41, 42, 45, 50, 51, 52, 53, 55, 57, 58, 60, 61, 62, 63, 64, 66, 67, 71, 73, 74, 75, 76, 77, 78, 79, 80, 81, 83, 87, 91, 94, 97, 98, 99.

e'_i : 0, 1, 2, 3, 4, 8, 11, 14, 16, 19, 20, 25, 27, 28, 29, 30, 35, 37, 39, 40, 44, 47, 48, 50, 53, 54, 57, 58, 60, 61, 62, 66, 68, 69, 70, 71, 73, 76, 78, 79, 83, 85, 89, 91, 94.

Now, if E is defined from (4) as in Theorem 2, so that the e_i's less than 100 are as above, then

$$\nu(S(n, 5)) = -1 + \nu(n - E) \text{ for even } n, \text{ provided } \nu(n-E) < n - 1,$$

and similarly for odd n. The first failure of this would be for n equal to the smallest $E(k)$ such that $e_{k+1} \geq E(k) - 1$. It can be observed that this never happens for $5 \leq n < 2^{100}$, and for it to happen subsequently would require the unlikely occurrence of more than 2^{94} consecutive 0's in the binary expansion of E or E'.

It is then a simple matter to read off $\nu(S(n, 5))$ for n even and $n < 2^{100}$ from the smallest 2-power in which n differs from E, as determined from the list of e_i's, and similarly for n odd. For example, since

$$1989 = 2^0 + 2^2 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10},$$

we obtain $\nu(S(1989, 5)) = -1 + \nu(1989 - E') = 0$.

A similar discussion to all of this applies to $S(n, 6)$. If $k < 5$, then $\nu(S(n, k)) = 0$ or 1 depending on the parity of n, and for $k > 7$, $\nu(S(n, k))$ is somewhat more complicated to analyze. In work [1] stimulated by an earlier version of this paper, Francis Clarke has generalized this work in a number of directions (larger k, odd primes, and a more general context). I thank him for several useful comments on this work.

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