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# ON BLOCH'S CONSTANT

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ABSTRACT. The lower bound for Bloch's constant is slightly improved.

Let $H(\mathbf{D})$ denote the class of holomorphic functions in $\mathbf{D} := \{z \in \mathbf{C}: |z| < 1\}$. Given a function $F \in H(\mathbf{D})$ define $B_F$ to be the least upper bound of all numbers $r > 0$ such that there exists a number $z_0 \in \mathbf{C}$ and a region $\Omega \subseteq \mathbf{D}$ which is univalently mapped onto $\{z \in \mathbf{C}: |z - z_0| < r\}$ by $F$. Now Bloch's constant $B$ is

$$B := \inf\{B_F: F \in H(\mathbf{D}) \wedge F'(0) = 1\}.$$

The purpose of this note is to give a new proof of the well-known inequality $B > \sqrt{3}/4$ (cf. [3, 5, 8]). Indeed our method of proof enables us to show $B > \sqrt{3}/4 + 10^{-14}$. We need the following

**Theorem 1.** $B = \inf\{B_F: F \in \mathscr{B} \wedge F'(0) = 1\}$ (cf. [4]).

Here $\mathscr{B}$ denotes the class of all functions $F \in H(\mathbf{D})$ with

$$F(0) = 0 \quad \text{and} \quad \sup_{z \in \mathbf{D}} |F'(z)|(1 - |z|^2) \leq 1.$$

Now we can prove

**Theorem 2.** *Let* $F \in \mathscr{B}$ *and* $F'(0) = 1$. *Then*

$$\operatorname{Re} F'(z) \geq \frac{1 - \sqrt{3}|z|}{(1 - \sqrt{1/3}|z|)^3} \quad \text{for } |z| \leq \sqrt{1/3}.$$

*Proof.* Define

$$f(w) := \frac{\sqrt{3}}{3} \frac{1 - w}{1 - w/3}$$

and

$$G(w) := \tfrac{9}{4} w \left(1 - \tfrac{1}{3} w\right)^2.$$

Calculation yields

$$|G(w)|(1 - |f(w)|^2) = 1 \quad \text{for } w \in \partial \mathbf{D}.$$

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Thus if $F \in \mathscr{B}$,

$$\left| \frac{F'(f(w))}{G(w)} \right| \leq 1 \quad \text{for } w \in \partial \mathbf{D}. \tag{1}$$

Now let

$$H(w) := \left[ \frac{F'(f(w))}{G(w)} - 1 \right] \frac{w}{(w-1)^2}.$$

If $F \in \mathscr{B}$ and $F'(0) = 1$, then

$$|F'(z)| = |1 + F''(0)z + \cdots| \leq \frac{1}{1 - |z|^2} = 1 + |z|^2 + \cdots ,$$

and therefore $F''(0) = 0$. Considering local developments in $w = 1$ we see that $H$ is a holomorphic function in $\overline{\mathbf{D}}$. Now (1) gives

$$\operatorname{Re} H(w) \geq 0 \quad \text{for } w \in \partial \mathbf{D},$$

and so

$$\operatorname{Re} H(w) \geq 0 \quad \text{for } w \in \overline{\mathbf{D}}.$$

Hence we get

$$\operatorname{Re} F'(f(w)) \geq G(w) \quad \text{for } w \in [0, 1].$$

This is equivalent to the assertion, for without loss of generality we may assume $z \in [0, \sqrt{1/3}]$.

**Corollary 1.** $B \geq \sqrt{3}/4$.

*Proof*. If $F \in \mathscr{B}$ and $F'(0) = 1$, we see from Theorem 2 that $\operatorname{Re} F'(z) > 0$ for $|z| < \sqrt{1/3}$. Hence $F$ is injective in $M := \{z \in \mathbf{C} \colon |z| \leq \sqrt{1/3}\}$. The inequality

$$|F(\sqrt{1/3}e^{i\varphi})| \geq \int_0^{\sqrt{1/3}} \operatorname{Re} F'(te^{i\varphi})\, dt \geq \int_0^{\sqrt{1/3}} \frac{1 - \sqrt{3}t}{(1 - \sqrt{1/3}t)^3}\, dt = \sqrt{3}/4$$

shows that every boundary point $F(\sqrt{1/3}e^{i\varphi})$, $\varphi \in [-\pi, \pi]$, of $F(M)$ has a distance to the origin which is greater than or equal to $\sqrt{3}/4$. Because $0 = F(0) \in F(M)$, $F(M)$ contains a disc of radius $\sqrt{3}/4$. This means $B_F \geq \sqrt{3}/4$ and so Theorem 1 shows $B \geq \sqrt{3}/4$.

To improve this result we need some preparation.

**Lemma 1.** *If* $F \in \mathscr{B}$ *and if* $F'$ *has the Taylor series*

$$F'(z) = 1 + \sum_{n=2}^{\infty} a_n z^n,$$

*then*

$$|a_2| \leq 1, \qquad |a_3| \leq 5$$

*and*

$$\sum_{n=4}^{\infty} |a_n||z|^n \le 10|z|^4 \quad \textit{for } |z| \le 1/10.$$

*Proof.* $|a_2| \le 1$ follows at once from

$$|F'(z)| = |1 + a_2 z^2 + \cdots| \le 1/(1-|z|^2) = 1 + |z|^2 + \cdots.$$

If $0 < r < 1$, then

$$1 + \sum_{n=2}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |F'(re^{i\theta})|^2\, d\theta \le \frac{1}{(1-r^2)^2}.$$

This implies

$$|a_n|^2 \le \frac{1}{r^{2n}}\left[\frac{1}{(1-r^2)^2} - 1\right] \quad \text{for } n \ge 3,$$

and thus

$$|a_n| \le \frac{\sqrt{2-r^2}}{r^{n-1}(1-r^2)} \quad \text{for } n \ge 3.$$

For $n = 3$ and $r = \sqrt{3/5}$ we immediately obtain

$$|a_3| \le 5.$$

For $n \ge 4$ set $r = \sqrt{(n-1)/(n+1)}$. For $n \ge 2$ the inequality

$$\left(\frac{n+1}{n-1}\right)^{(n-1)/2} = \left(1 + \frac{2}{n-1}\right)^{(n-1)/2} \le e$$

is valid, because the function $(1 + 1/x)^x$ in $(0, \infty)$ is monotonically increasing and tends to $e$ for $x \to \infty$. So we arrive at

$$|a_n| \le \frac{n+1}{2}\left(\frac{n+1}{n-1}\right)^{(n-1)/2}\sqrt{\frac{n+3}{n+1}} \le \sqrt{\frac{7}{5}}e\frac{n+1}{2}.$$

This yields, for $|z| \le 1/10$,

$$\begin{aligned}\sum_{n=4}^{\infty} |a_n||z|^n &\le \sum_{n=4}^{\infty} \sqrt{\frac{7}{5}}e\frac{n+1}{2}|z|^n = \frac{5}{2}\sqrt{\frac{7}{5}}e\sum_{n=4}^{\infty}\frac{n+1}{5}|z|^n \\ &\le \frac{5}{2}\sqrt{\frac{7}{5}}e|z|^3\sum_{n=1}^{\infty} n|z|^n = \frac{5}{2}\sqrt{\frac{7}{5}}e\frac{|z|^4}{(1-|z|)^2} \le 10|z|^4.\end{aligned}$$

**Lemma 2.** *Let* $F \in \mathscr{B}$ *with* $F'(0) = 1$, $\varphi_0 = 2\arcsin(1/20)$ *and* $|z| \le 1/1000$. *Then the following inequality is valid for* $\varphi \in [-\pi, \pi] \setminus [-\varphi_0, \varphi_0]$:

$$\frac{1}{2}\mathrm{Re}[F'(z) + F'(e^{i\varphi}z)] \ge \frac{1-\sqrt{3}|z|}{(1-\sqrt{1/3}|z|)^3} + \frac{1}{2}|z|^3.$$

*Proof.* Since $F \in \mathscr{B}$ and $F'(0) = 1$ the function $F'$ has a Taylor series whose second coefficient vanishes:

$$F'(z) = 1 + \sum_{n=2}^{\infty} a_n z^n.$$

Lemma 1 shows that, for $\varphi \in [-\pi, \pi]$ and $|z| \le 1/10$,

$$\begin{aligned}
&\frac{1}{2}\operatorname{Re}[F'(z) + F'(e^{i\varphi}z)] - \frac{1-\sqrt{3}|z|}{(1-\sqrt{1/3}|z|)^3} \\
&\quad = \operatorname{Re}\left[\sum_{n=2}^{\infty}\frac{1}{2}(1+e^{in\varphi})a_n z^n\right] + \sum_{n=2}^{\infty}\frac{n^2-1}{\sqrt{3}^n}|z|^n \\
&\quad \ge \operatorname{Re}\left[e^{i\varphi}a_2 z^2\cos\varphi + e^{i(3/2)\varphi}a_3 z^3\cos\frac{3}{2}\varphi\right] + |z|^2 + \frac{8\sqrt{3}}{9}|z|^3 - 10|z|^4 \\
&\quad \ge (1-|\cos\varphi|)|z|^2 + \left(\frac{8\sqrt{3}}{9} - 5\left|\cos\frac{3}{2}\varphi\right|\right)|z|^3 - 10|z|^4.
\end{aligned}$$

Now if $\varphi \in [-\pi, -\pi+\varphi_0]$ or $\varphi \in [\pi-\varphi_0, \pi]$ set $\alpha = (\pi+\varphi)/2$ resp. $\alpha = (\pi-\varphi)/2$. Then $\alpha \in [0, \varphi_0/2]$ and we obtain the following lower bound for the expression above:

$$\begin{aligned}
\left(\frac{8\sqrt{3}}{9} - 5\sin 3\alpha\right)|z|^3 - 10|z|^4 &\ge \left(\frac{8\sqrt{3}}{9} - 15\sin\alpha\right)|z|^3 - 10|z|^4 \\
&\ge \left(\frac{8\sqrt{3}}{9} - \frac{3}{4}\right)|z|^3 - 10|z|^4 \ge \tfrac{3}{4}|z|^3 - 10|z|^4.
\end{aligned}$$

If on the other hand $\varphi \in [-\pi+\varphi_0, -\varphi_0] \cup [\varphi_0, \pi-\varphi_0]$, we obtain the lower bound

$$(1-\cos\varphi_0)|z|^2 - 4|z|^3 - 10|z|^4 = \tfrac{1}{200}|z|^2 - 4|z|^3 - 10|z|^4.$$

Now for $|z| \le 1/1000$ both

$$\tfrac{3}{4}|z|^3 - 10|z|^4 \ge \tfrac{1}{2}|z|^3$$

and

$$\tfrac{1}{200}|z|^2 - 4|z|^3 - 10|z|^4 \ge \tfrac{1}{2}|z|^3.$$

Thus we get the inequality

$$\frac{1}{2}\operatorname{Re}[F'(z) + F'(e^{i\varphi}z)] - \frac{1-\sqrt{3}|z|}{(1-\sqrt{1/3}|z|)^3} \ge \frac{1}{2}|z|^3,$$

which is valid for $\varphi \in [-\pi, \pi] \setminus [-\varphi_0, \varphi_0]$ and $|z| \le 1/1000$.

Now we can prove:

**Corollary 2.** $B > \sqrt{3}/4 + 10^{-14}$.

*Proof*. It suffices to show that every function $F \in \mathscr{B}$ with $F'(0) = 1$ covers a disc of radius $\sqrt{3}/4 + \frac{1}{13}10^{-12}$ univalently.

If

$$\min_{\varphi\in[-\pi,\pi]} \operatorname{Re}[e^{-i\varphi}F(\sqrt{1/3}e^{i\varphi})] \ge \tfrac{\sqrt{3}}{4} + \tfrac{1}{13}10^{-12},$$

then the assertion follows from the same arguments as those in the proof of Corollary 1.

If on the other hand the above minimum is less than $\sqrt{3}/4 + \frac{1}{13}10^{-12}$, we can without loss of generality assume that it is obtained for $\varphi = 0$. Letting $\varphi_0 = 2\arcsin(1/20)$ Lemma 2 shows, for $\varphi \in [-\pi, \pi] \setminus [-\varphi_0, \varphi_0]$,

$$\begin{aligned}
&\tfrac{1}{2}\operatorname{Re}[F(\sqrt{1/3}) + e^{-i\varphi}F(\sqrt{1/3}e^{i\varphi})] \\
&\quad = \frac{1}{2}\operatorname{Re}\left[\int_0^{\sqrt{1/3}} (F'(r) + F'(e^{i\varphi}r))\,dr\right] \\
&\quad = \int_0^{\sqrt{1/3}} \frac{1}{2}\operatorname{Re}[F'(r) + F'(e^{i\varphi}r)]\,dr = \int_0^{1/1000} \cdots + \int_{1/1000}^{\sqrt{1/3}} \cdots \\
&\quad \geq \int_0^{1/1000} \frac{1}{2}r^3\,dr + \int_0^{\sqrt{1/3}} \frac{1 - \sqrt{3}r}{(1 - \sqrt{1/3}r)^3}\,dr = \frac{\sqrt{3}}{4} + \frac{1}{8}10^{-12},
\end{aligned}$$

hence

$$\operatorname{Re}[e^{-i\varphi}F(\sqrt{1/3}e^{i\varphi})] > \tfrac{\sqrt{3}}{4} + \tfrac{9}{52}10^{-12}.$$

For $\varphi \in [-\varphi_0, \varphi_0]$ we have the following inequality which follows at once from Theorem 2:

$$\operatorname{Re}[e^{-i\varphi}F(\sqrt{1/3}e^{i\varphi})] \geq \tfrac{\sqrt{3}}{4}.$$

Now let $z_0 := -\frac{5}{52}10^{-12}$ and $r_0 := \sqrt{3}/4 + \frac{1}{13}10^{-12}$. Then

$$\{z \in \mathbf{C}\colon |z - z_0| < r_0\} \cap F(\{z \in \mathbf{C}\colon |z| = \sqrt{1/3}\}) = \emptyset,$$

because for $\varphi \in [-\pi, \pi] \setminus [-\varphi_0, \varphi_0]$ we have

$$\begin{aligned}
|F(\sqrt{1/3}e^{i\varphi}) - z_0| &= |e^{-i\varphi}F(\sqrt{1/3}e^{i\varphi}) - e^{-i\varphi}z_0| \\
&\geq \operatorname{Re}[e^{-i\varphi}F(\sqrt{1/3}e^{i\varphi})] - |z_0| \\
&> \tfrac{\sqrt{3}}{4} + \tfrac{9}{52}10^{-12} - \tfrac{5}{52}10^{-12} = r_0,
\end{aligned}$$

and for $\varphi \in [-\varphi_0, \varphi_0]$ we have

$$\begin{aligned}
|F(\sqrt{1/3}e^{i\varphi}) - z_0| &= |e^{-i\varphi}F(\sqrt{1/3}e^{i\varphi}) - e^{-i\varphi}z_0| \\
&\geq \operatorname{Re}[e^{-i\varphi}F(\sqrt{1/3}e^{i\varphi}) - e^{-i\varphi}z_0] \\
&\geq \tfrac{\sqrt{3}}{4} + \tfrac{5}{52}10^{-12}\cos\varphi > \tfrac{\sqrt{3}}{4} + \tfrac{1}{13}10^{-12} = r_0.
\end{aligned}$$

Thus in all cases $F$ covers a disc of radius $r_0 = \sqrt{3}/4 + \frac{1}{13}10^{-12}$ univalently. This means $B_F \geq \sqrt{3}/4 + \frac{1}{13}10^{-12}$.

*Remarks.* Theorem 2 also gives the lower bound $S \geq \sqrt{1/3}$ for the Marden constant $S$ for Bloch functions which appears in [6].

A systematic treatment of the ideas developed in this paper is given in [2]. The main result is the determination of the variability regions

$$V(z_1, w_1; z_2) := \{F'(z_2)\colon F \in \mathscr{B} \wedge F'(z_1) = w_1\}$$

for $z_1$, $z_2 \in \mathbf{D}$ and $w_1 \in \mathbf{C}$.

## REFERENCES

1. L. V. Ahlfors, *An extension of Schwarz's lemma*, Trans. Amer. Math. Soc. **43** (1938), 359–364.
2. M. Bonk, *Extremalprobleme bei Bloch-Funktionen*, Dissertation Braunschweig, 1988.
3. M. Heins, *On a class of conformal metrics*, Nagoya Math. J. **21** (1962), 1–60.
4. E. Landau, *Über die Bloch'sche Konstante und zwei verwandte Weltkonstanten*, Math. Z. **30** (1929), 608–634.
5. C. D. Minda, *Bloch constants*, J. Analyse Math. **41** (1982), 54–84.
6. ——, *Marden constants for Bloch and normal functions*, J. Analyse Math. **42** (1982/83), 117–127.
7. E. Peschl, *Les invariants differentiels non holomorphes et leur rôle dans la théorie des fonctions*, Rend. Sem. Math. Messina **1** (1955), 100–108.
8. Ch. Pommerenke, *On Bloch functions*, J. London Math. Soc. (2) **2** (1970), 689–695.

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