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EXAMPLE OF AN ALGEBRA WHICH IS NONTOPOLOGIZABLE AS A LOCALLY CONVEX TOPOLOGICAL ALGEBRA

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ABSTRACT. Let X be a real or complex linear space and denote by $L(X)$ the algebra of all its endomorphisms. We prove that $L(X)$ is topologizable as a locally convex topological algebra (with jointly continuous multiplication) if and only if it is topologizable as a Banach algebra and this holds if and only if X is of finite dimension.

A topological algebra is a Hausdorff topological linear space equipped with a jointly continuous associative multiplication. A topological algebra is said to be locally convex if its underlying topological linear space is locally convex. The topology of a locally convex algebra can be given by means of a family $(\|x\|_\alpha)$ of seminorms such that, for each index α, there is an index β such that

$$\|xy\|_\alpha \leq \|x\|_\beta \|y\|_\beta \tag{1}$$

for all x and y in the algebra in question (see [4]). For general information on topological algebras the reader is referred to [1, 2, 3, 4] and the references therein. Note that some authors define topological algebras as topological linear spaces equipped with a separately continuous multiplication (cf. [2]).

In [5] and [6] we posed the following question (Problem 2): Is it true that for every real or complex algebra there is a topology making it a locally convex algebra? In [6] we have shown that the answer is positive if we replace the requirement of joint continuity of multiplication by the weaker assumption of its separate continuity. We also conjectured (cf. [6, Problem 2a]) that the answer to the problem is negative if we take, as the algebra in question, the algebra of all endomorphisms of an infinite-dimensional real or complex linear space. The aim of this paper is to prove this conjecture, which gives the nontrivial implication in the result formulated in the abstract.

Theorem. *Let X be a real or complex infinite-dimensional vector space. Then there is no topology on the algebra $L(X)$ of all endomorphisms of X making it a locally convex algebra.*

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Proof. Assume that there is a system $(\|x\|_\alpha)$ of seminorms on $L(X)$ satisfying relations (1) and separating between the points of $L(X)$, and try to get a contradiction. Consider on $L(X)$ the one-dimensional operators $f \otimes z$, given by $x \to f(x)z$, where f is a linear functional on X and $z \in X$. Let g_0 be a fixed linear functional on X, $g_0 \neq 0$, and fix a nonzero element z_0 in X. Since the seminorms on $L(X)$ separate between its elements we can choose an index α_0 so that $\|g_0 \otimes z_0\|_{\alpha_0} \neq 0$. After a suitable normalization of g_0 we can assume

$$\|g_0 \otimes z_0\|_{\alpha_0} = 1. \tag{2}$$

For an arbitrary functional f on X and an arbitrary element x in X, the operators $f \otimes z_0$ and $g_0 \otimes x$ are in $L(X)$. Applying to these operators relations (1) with $\alpha = \alpha_0$ and a suitable β_0, we obtain

$$\|(f \otimes z_0)(g_0 \otimes x)\|_{\alpha_0} \leq \|f \otimes z_0\|_{\beta_0} \|g_0 \otimes x\|_{\beta_0}. \tag{3}$$

We have $(f \otimes z_0)(g_0 \otimes x) = f(x) g_0 \otimes z_0$, and so, by (2), the left hand of (3) is $|f(x)|$. Put $c(f) = \|f \otimes z_0\|_{\beta_0}$ and $\|x\| = \|g_0 \otimes x\|_{\beta_0}$. With this notation we can rewrite (3) as

$$|f(x)| \leq c(f)\|x\|. \tag{4}$$

Relation (4) holds for all linear functionals f on X and all elements x in X; it implies that $\|\cdot\|$ is a norm on X. This is because $\|\cdot\|$ is a seminorm on X and, for any nonzero element x in X, there is a linear functional f on X with $f(x) \neq 0$. Thus X equipped with the norm $\|\cdot\|$ is a normed space and, by (4), all its linear functionals are continuous. It is obvious that the only normed spaces with this property are finite-dimensional and X is of infinite dimension. The conclusion follows. □

Corollary. *If X is an infinite-dimensional real or complex vector space, then there is no topology on the algebra $L_{FD}(X)$ of all finite-dimensional endomorphisms of X making of it a locally convex algebra.*

The problem still remains open as to whether every commutative algebra over real or complex scalars can be given a topology so that it becomes a locally convex algebra.*

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* Added in proof. In a letter to the author Vladimir Müller gives a construction of a commutative algebra which is nontopologizable as a topological algebra. He shows also that the above theorem is true if we replace "locally convex algebra" by "topological algebra."

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