A RESULT ABOUT THE HILBERT TRANSFORM ALONG CURVES

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Abstract. Let $G$ be a connected and simply connected, nilpotent Lie group and let $\gamma: (-1, 1) \to G$ be a (connected) analytic curve such that $\gamma(0) = 0$. Then the Hilbert transform along $\gamma$,

$$Tf(x) = \text{p.v.} \int_{0<|t|<1} \frac{f(xy(t)^{-1})}{t} dt,$$

is bounded on $L^p(G)$, $1 < p < \infty$.

Introduction

Let $G$ be a simply connected, nilpotent Lie group. From [Ch] we know that if $\gamma$ is a homogeneous curve with respect to a group of dilations of $G$, $\gamma(0) = 0$, then the Hilbert transform along $\gamma$ is a bounded operator on $L^p(G)$, for $1 < p < \infty$; moreover $Tf(x)$ exists a.e. for all $f \in L^p(G)$. In [RS2] singular integral operators whose kernels are homogeneous distributions of critical degree with mean value zero and supported on a homogeneous, analytic submanifold of $G$ are considered.

The purpose of this note is to prove that if $\gamma: (-1, 1) \to G$ is a connected, analytic curve such that $\gamma(0) = 0$ then the

$$Tf(x) = \text{p.v.} \int_{0<|t|<1} \frac{f(xy(t)^{-1})}{t} dt,$$

is a bounded operator on $L^p(G)$, $1 < p < \infty$.

For the proof we first assume that a certain group of dilations associated to $\gamma$ are automorphisms of $G$ and also that $\gamma$ generates $G$. We use the iteration argument of [Ch]. The general case follows from a transference theorem [RS2]. Finally the boundedness of the maximal operator

$$T^*f(x) = \sup_{r>0} \left| \int_{r<|t|<1} \frac{f(xy(t)^{-1})}{t} dt \right|$$

on $L^p(G)$, implies that $Tf(x)$ exists a.e. for all $f \in L^p$.
The proof of the theorem. Let \( G \) be a connected and simply connected, nilpotent Lie group. Let \( \gamma : (-1, 1) \to G \) be a (connected) analytic curve that generates \( G \) in the sense that \( \gamma \) is not contained in any proper closed subgroup of \( G \). Also assume that \( \gamma(0) \) is the identity of \( G \). Denote by \( \exp : \mathfrak{g} \to G \) the exponential map. First of all we remark the following facts:

(i) \( \gamma \) generates \( G \) if and only if \( \{ d^j \gamma/dt^j \}_{j \in \mathbb{N}} \) generate \( \mathfrak{g} \) as a Lie algebra, with \( \tilde{\gamma} = \gamma \circ \exp^{-1} \). Indeed, let \( \mathfrak{h} \) be the Lie subalgebra of \( \mathfrak{g} \) generated by \( \{ d^j \gamma/dt^j \}_{j=0} \) and \( H = \exp \mathfrak{h} \). Since \( \gamma \) is analytic, \( \tilde{\gamma} \subset \mathfrak{h} \).

(ii) We can choose a coordinate system on \( G \) such that

\[
\gamma(t) = \left( \frac{t^{a_1}}{a_1!} \varphi_1(t), \ldots, \frac{t^{a_k}}{a_k!} \varphi_k(t), 0, \ldots, 0 \right)
\]

with \( 1 \leq a_1 < a_2 < \cdots < a_k \), \( \varphi_i \) are analytic functions and \( \varphi_i(0) = 1 \). Indeed, following [SW], we choose \( a_1 = \inf \{ j | d^j \gamma/dt^j \neq 0 \} \). Given \( a_1, \ldots, a_i \), define

\[
a_{i+1} = \inf \left\{ j | j > a_i \text{ and } \left\{ \frac{d^i \gamma}{dt^i} \Bigr|_{t=0}, \ldots, \left. \frac{d^j \gamma}{dt^j} \Bigr|_{t=0}, \frac{d^j \gamma}{dt^j} \Bigr|_{t=0} \right\} \right\}
\]

are linearly independent.

Define \( e_i = d^i \gamma/dt^i \Bigr|_{t=0} \). We thus obtain a set \( \{ e_1, \ldots, e_k \} \), \( k \leq n \), maximal with respect to be L.I. such that

\[
\tilde{\gamma}(t) = \left( \frac{t^{a_1}}{a_1!} \varphi_1(t) e_1 + \cdots + \frac{t^{a_k}}{a_k!} \varphi_k(t) e_k. \right)
\]

We extend \( \{ e_1, \ldots, e_k \} \) to a basis of \( \mathfrak{g} \), \( \{ e_1, \ldots, e_k, e_{k+1}, \ldots, e_n \} \). If \( A \) denotes the change of basis matrix, the coordinates of \( \gamma(t) \) with respect to \( \psi = \exp \circ A \) are \( ((t^{a_1}/a_1!) \varphi_1(t), \ldots, (t^{a_k}/a_k!) \varphi_k(t), 0, \ldots, 0) \). From now on we fix this coordinate system on \( G \).

(iii) We now associate to \( \gamma \) a group of dilations \( D_r, \ r > 0 \), by

\[
D_r(x_1, \ldots, x_n) = (r^{a_1} x_1, \ldots, r^{a_k} x_k, r^{a_{k+1}} x_{k+1}, \ldots, r^{a_n} x_n),
\]

where \( a_1, \ldots, a_k \) are as in (ii) and \( a_{k+1}, \ldots, a_n \) are arbitrary integers. Let \( Q = a_1 + \cdots + a_n \).

(iv) Let \( \gamma_0 \) be the curve in \( G \) with coordinates

\[
(t^{a_1}/a_1!, \ldots, t^{a_k}/a_k!, 0, \ldots, 0).
\]

Then \( \gamma_0 \) is analytic, homogeneous with respect to \( D_r \) and generates \( G \). Indeed, if \( \gamma = \psi^{-1} \circ \gamma \) and \( \gamma_0 = \psi^{-1} \circ \gamma \), then \( d^i \gamma_0/dt^i \Bigr|_{t=0} = d^i \gamma/dt^i \Bigr|_{t=0} = e_i, \ 1 \leq i \leq k \).

We have associated to \( \gamma \) a group of dilations \( \{ D_r \}_{r>0} \) and a curve \( \gamma_0 \) that is homogeneous with respect to \( D_r \) and also generates \( G \). For the moment we also assume that \( \{ D_r \} \) are automorphisms of \( G \) and fix a homogeneous norm.
with respect to $D_r$, e.g., $|x| = \max_{1 \leq i \leq n} \{|x_i|^{1/a_i}\}$. Denote by $\bar{B}$ the closure of the unit ball in $\mathbb{R}^n$.

**Lemma 1.** Let $\phi, \phi_0$ be the functions on $\bar{B}$ defined by $\phi(t_1, \ldots, t_n) = \gamma(t_1) \cdots \gamma(t_n)$; $\phi_0(t_1, \ldots, t_n) = \gamma_0(t_1) \cdots \gamma_0(t_n)$. For each $j \in \mathbb{N}$, $j \geq 0$, define $\phi_j$ on $\{t \in \mathbb{R}^n : \frac{1}{2} \leq |t_j| \leq 1\}$ by $\phi_j(t) = D_2H(\phi(t^j))$. If $J_j(t) = \det d\phi_j(t)$ and $J_0(t) = \det d\phi_0(t)$, then $J_j(t) \to J_0(t)$ uniformly on $\{t \in \mathbb{R}^n : \frac{1}{2} \leq |t_j| \leq 1\}$.

**Proof.** Let $\phi_j(t) = \exp^{-1} \phi_j(t), \phi_0(t) = \exp^{-1} \phi_0(t)$.

$$\phi_j(t) = \exp^{-1}(D_2H(\phi(t^j))) = D_2H(\phi(t^j))$$

$$= D_2H(\phi(t^j) \cdots \phi(t_n)) = \sum_{j(k) \in \mathcal{J}} c_{j(k)}(t^j) \cdots c_{j(k)}(t_n).$$

Then

$$\phi_j(t) = \sum_{j(k) \in \mathcal{J}} c_{j(k)}(t^j) \cdots c_{j(k)}(t_n) + \phi_0(t).$$

If $\gamma$ is homogeneous, $\phi_j(t) = \phi(t)$ and the first term of the right side of (*) is also independent of $j$. Thus either $P_j \equiv 0$ or $P_j(t_1, \ldots, t_n)$ is homogeneous of degree $a_j$, $j = 1, \ldots, n$,

$$P_j(t_1, \ldots, t_n) = \sum_{j(k) \in \mathcal{J}} c_{j(k)}(t^j) \cdots c_{j(k)}(t_n)$$

with $a_{j_1} + \cdots + a_{j_k} = a_j$, $b_{j_1} \cdots b_{j_k} = \mathfrak{C}$ and $F_{j_1}, \ldots, j_k(0) = 1$. Then

$$\phi_j(t) = \left( \sum_{i=1}^{n} \phi_i(2^{-j}t_i), \ldots \right)$$

By the mean value theorem applied to $\varphi_i$ and $F_{j_1}, \ldots, j_k$, we have that

$$\phi_j(t) = \left( \sum_{i=1}^{n} \phi_i(2^{-j}t_i), \ldots \right)$$

with $G_k$ bounded on $\{t_j \leq |t_j| \leq 1\}, k = 1, \ldots, n$. This proves the lemma since $\det d_x \exp = 1$.

**Remark.** (v) If $\gamma$ is an analytic curve that generates $G$ then $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(t_1, \ldots, t_n) = \gamma(t_1) \cdots \gamma(t_n)$ has rank $n$ almost everywhere. Indeed, since $\gamma_0$ is a homogeneous curve that generates $G$, $J_0 \neq 0$ [Ch]. Since $\gamma_0$ is analytic,
$J_0(t) \neq 0$ almost everywhere. Let $J(t) = \det \phi(t)$. Since $\phi_j(t) = D_{2j} \phi(2^{-j} t)$, $2^{j(Q-n)} J(2^{-j} t) = J_j(t) \to J_0(t)$ on $\{ t_1^{1/2} \leq |t_j| \leq 1 \}$. Then $J \neq 0$. (Furthermore $J(t) \neq 0$ almost everywhere).

By a result of [RS2] we know that if $\phi: \overline{B} \to \mathbb{R}^n$ is an analytic function having rank $n$ at almost everywhere point and $\psi$ is a $C^1$-function supported on $\overline{B}$ then the transported measure by $\phi$ of $\psi(t) \, dt$ is absolutely continuous and its density $\rho$ satisfies and $L^1$-Hölder condition $\int |\rho(x + y) - \rho(x)| \, dx \leq c\|y\|^{\delta}$, for some $\delta > 0$ depending on $\phi$, $c > 0$ that depends also on $\psi$.

In the next lemma we say a little more about $c$.

**Lemma 2.** Let $J(t) = \det D\phi(t)$. Then

$$
\int |\rho(x + y) - \rho(x)| \, dx \leq \tilde{C}\|y\|^{\delta} \left( \int_B |\nabla \psi| + |\psi| \right)^\delta \left( \int_B |\psi|/2\delta/J(1 - \delta) \right)^{1-\delta},
$$

for some $\delta > 0$, $\tilde{C} > 0$, depending only on $\phi$.

**Proof.** Following the proof of [RS2] we denote by $Z$ the set of zeros of $J(t)$ and consider a covering $\{ B_j(z_j, r_j) \}$ of $\overline{B} - Z$, satisfying,

1. $r_j = c_0 |J(z_j)|$;
2. the balls $B^*_j = B(z_j, 2r_j)$ are disjoint from $Z$ and have the bounded overlapping property; i.e. there is an integer $N$ such that no point belongs to more than $N$ of the $B^*_j$.

Let $\{ n_j \}$ be a smooth partition of unity on $\overline{B} - Z$ subordinated to $\{ B_j \}$ and such that $\|\nabla n_j\|_{\infty} \leq \tilde{C}r_j^{-1}$. Let $\psi_j = \psi n_j$ and let $\rho_j$ be the transported measure by $\phi$ of $\psi_j$. Since $\phi^\prime|_{B_j}$ is invertible [RS2] $\rho_j(x) = |J\phi^\prime(x)|^{-1} \psi_j(\phi^\prime(x))$ and thus

$$
\int |\rho_j(x)| \, dx = \|\psi_j\|_1 = \int |\psi_j|
$$

and

$$
\int |\nabla \rho_j(x)| \, dx \\
\leq \tilde{C} \left( \int |\nabla \psi(t)|n_j(t)|J(t)|^{-1} \, dt + \int |\psi| |\nabla n_j| |J(t)|^{-1} \, dt + \int |J(t)|^{-2} |\psi| \, dt \right) \\
\leq \tilde{C}r_j^{-2} \left( \int_{B_j} |\nabla \psi| + \int_{B_j} |\psi| \right).
$$

This implies that if $0 < \delta \leq 1$,

$$
\int |\rho_j(x + y) - \rho_j(x)| \, dx \leq \tilde{C}\|y\|^{\delta} r_j^{-2\delta} \left( \int_{B_j} |\nabla \psi| + |\psi| \right)^\delta \left( \int_{B_j} |\psi| \right)^{1-\delta}.
$$
Now by Hölder's inequality
\[
\sum_j \left( \int_{B_j} |\nabla \psi| + |\psi| \right)^\delta r_j^{-2\delta} \left( \int_{B_j} |\psi| \right)^{1-\delta} 
\leq \left( \sum_j \left( \int_{B_j} |\nabla \psi| + |\psi| \right) \right)^\delta \left( \sum_j r_j^{-2\delta/(1-\delta)} \int_{B_j} |\psi| \right)^{1-\delta}.
\]

The lemma follows from the fact that \( J |B^* \) is comparable with \( r_j \) and from the bounded overlapping property.

**Theorem.** Let \( \gamma \) be an analytic curve that generates \( G \) and such that the group of dilations associated to \( \gamma \) are automorphisms of \( G \). Then
\[
Tf(x) = p.v. \int_{0<|t|<1} f(x\gamma(t))^{-1} dt/t
\]
defines a bounded operator on \( L^p(G), 1 < p < \infty \).

**Proof.** (We follow [RS2] for the proof.) Denoted by \( \mu \) the distribution given by \( \mu(f) = p.v. \int_{0<|t|<1} f(\gamma(t)) dt/t \). Let \( \phi \) be in \( \mathcal{S}^\infty(1/2, 2) \) such that \( \sum_{j=0}^{\infty} \phi(2^j|t|) = 1 \). Let \( \varphi(x) = \varphi(|x|) \) where \( | \cdot | \) denote the homogeneous norm defined in Remark (iv). Let \( \mu_j = \phi(2^j|t|) \mu \).

Let \( \varphi_j(x) = 2^j \varphi(D_{2^j}x) \) and \( \eta_j = \frac{1}{c} (\varphi_{j+1} - \varphi_j) \) with \( c = \int \varphi \).

Since \( \{c^{-1}\varphi_j\} \) is an approximate identity in \( G \) we can write
\[
\delta = c^{-1} \lim_{j \to +\infty} \varphi_j = \sum_{j \geq j_0} \eta_j + c^{-1} \varphi_{j_0}
\]
for each \( j_0 \) fixed. Also \( \eta = \sum_{j \geq 0} \mu_j \). Then
\[
\mu = \delta \sum_j \mu_j = \sum_j \left( \sum_{k \geq j_0} \eta_k \ast \mu_j \right) + c^{-1} \sum_j \varphi_{j_0} \ast \mu_j.
\]
For each \( j \) fixed, take \( j_0 = j + 1 \). Then
\[
\mu = \sum_{k \geq 1} \left( \sum_{j \geq 0} \eta_{k+j} \ast \mu_j \right) + c^{-1} \sum_j (\varphi_{j+1} \ast \mu_j)
\]
\[
= \sum_{k \geq 1} M_k + N.
\]
If we prove
(1) For \( 0 < \epsilon \leq 1 \), \( \|M_k\|_{p,p} \leq C_{\epsilon,p} 2^{\epsilon k} \), \( \|N\|_{p,p} \leq C \).
(2) \( \exists \sigma > 0 \) such that \( \|M_k\|_{2,2} \leq C 2^{-\sigma k} \).

Then the theorem follows by interpolation and duality. Indeed, fix \( 1 < p < 2 \) and take \( 1 < p_0 < p \). Then \( \frac{1}{p} = \frac{1}{2} s + (1-s) \frac{1}{p_0} \) for some \( 0 < s < 1 \). Thus
\[ \|M_k\|_{p,p} \leq C_{\epsilon} 2^{-\sigma k} 2^{(k(1-\epsilon))} \] for \( 0 < \epsilon \leq 1 \) and some \( \sigma > 0 \). Choosing \( \epsilon \) small enough we obtain \( \sum_k \|M_k\|_{p,p} < +\infty \).

(1) We observe that \( M_k = \sum_{j \geq 0} \eta_{k+j} * \mu_j = \sum_{j \geq 0} f_j^k \) where \( f_j^k \) has the following properties

(a) \( f_j^k = 0 \),
(b) \( \int |f_j^k(xy) - f_j^k(x)| \, dx \leq C_2^{(k+j)\epsilon} |y|^{\epsilon} \),
(c) \( \sup f_j^k \subset \{x | |x| \leq C_2^{-l/1} \} \),
(d) \( \int |f_j^k| \leq C \).

Then Cotlar's lemma implies that \( \|M_k\|_{2,2} \leq C_2^{k} \) and the weak type (1.1) of \( M_k \) follows by checking that \( \int_{|x| \geq 2|y|} |M_k(xy) - M_k(x)| \, dx \leq C_2^{k}\epsilon \). The same argument holds for \( N \). For a proof see [ChNSW].

(2) By Cotlar's lemma it is enough to prove that for some \( \sigma > 0 \)

\[ \|f_j^k * f_l^k\|_{2,2} \leq C_2^{-\sigma k} 2^{-l(j-l)\sigma} \]. We check this for \( j > l \), by using the iteration argument of [Ch].

\[
\|f_j^k * f_l^k\|_{2,2} \leq C_2 \| \eta_{k+j} * \mu_j * \mu_l^* \|_{2,2} \\
\leq C_2 \| \eta_{k+j} * \mu_j \|_{1/2} \| \mu_l^* * \mu_l * \eta_{k+j} \|_{1/2} \\
\leq \ldots \leq C_2 \| \eta_{k+j} * \mu_j \|_{1-2^{-\sigma} \| \mu_l^* * \mu_l * \eta_{k+j} \|_{2^{-\sigma}}}. 
\]

Let \( \psi(t_1, \ldots, t_n) = \prod_{i=1}^n \frac{1}{t_i} \phi(2^i t_i) \) and \( \phi(t_1, \ldots, t_n) = \gamma(t_1) \cdots \gamma(t_n) \) as in Lemma 1.

Since \( \mu_1 * \cdots * \mu_j \) is the transported measure by \( \phi \) of \( \psi(t) \, dt \) and since \( \phi \) is analytic with rank \( n \) almost everywhere, \( \mu_1 * \cdots * \mu_j \) is absolutely continuous [RS2].

If \( \rho_l \) denotes its density, we have to prove that

\[ \| \rho_l * \mu_j^* * \eta_{k+j} \|_{1} \leq C_2^{-\sigma k} 2^{-l(j-l)\sigma} \] for some \( \sigma > 0 \). (Recall that we have assumed \( l < j \).)

Let \( \tilde{\rho}_l(x) = 2^{-lQ} \rho_l(D_2 - l x) \). Then \( \tilde{\rho}_l \) is the transported measure by \( \phi_l(t) = D_2 \phi(2^{-l} t) \) of \( \tilde{\psi}(t) = 2^{-lQ} \psi_l(2^{-l} t) \) (sup \( \tilde{\psi} \subset \{ t | \frac{1}{2} \leq |t| \leq 2 \} \)).

If we prove that

(A) \[ \int_{R^n} |\tilde{\rho}_l(xy) - \tilde{\rho}_l(x)| \, dx \leq C \| y \|^{\epsilon} \] for some \( C, \epsilon > 0 \), independent of \( l \)

then \( \int_{R^n} |\rho_l(xy) - \rho_l(x)| \, dx \leq C_2^{l/1} \| y \|^{\epsilon} \). From this and the fact that \( \eta_{k+j} \) has mean value zero and sop \( \eta_{k+j} \subset \{ x | |x| \leq C_2^{-(k+j)} \} \) we obtain the inequality desired.

But if \( \tilde{\rho}_l = \tilde{\rho}_l \circ \exp \) and \( \bar{\phi}_l = \exp^{-1} \circ \phi_l \) then \( \tilde{\rho}_l \) is the transported measure of \( \tilde{\psi}(t) \, dt \) by \( \bar{\phi}_l \) and (A) is equivalent to

(B) \[ \int_{R^n} |\tilde{\rho}_l(x+y) - \tilde{\rho}_l(x)| \, dx \leq C \| y \|^{\delta}, \] \( C, \delta > 0 \), independent of \( l \) (see [RS2]).
Finally we prove (B). Since by Lemma 2,
\[ \int_{\mathbb{R}^n} |\tilde{p}_f(x+y) - \tilde{p}_f(x)| \, dx \leq C \|y\|^\delta \left( \int |\nabla \tilde{\psi}| + |\tilde{\psi}| \right)^\delta \left( \int |\tilde{\psi}|/|J_\delta|^{2\delta/1-\delta} \right)^{1-\delta} \]
for some $0 < \delta < 1$, we only have to check that $\int_{\sup \tilde{\psi}} |J^{(l)}|^{-\alpha} \, dt \leq C$, with $C$ independent of $l$, for some $0 < \alpha < 1$. By Lemma 1, $2^{l(Q-n)} J(2^{-l} t) = J(t)$ converges uniformly to $J_0(t)$, which is a homogeneous polynomial of degree $Q - n$. Thus if we develop $J(t)$ about 0 in the Taylor expansion, $J(t) = P(t) + R(t)$, where $P$ is a homogeneous polynomial of degree $Q - n$ and $\frac{R(t)}{|t|^{Q-n}} \to 0$ as $|t| \to 0$. By a linear change of variables we can normalize $P$ in the $t_n$-direction, i.e., $P(t) = t_n^{Q-n} + \sum_{j=0}^{Q-n-1} b_j(t_1, \ldots, t_{n-1}) t_n^j$ with $b_j(0) = 0$. The Weierstrass preparation theorem now implies that
\[ J(t) = (t_1^{Q-n} + a_1(t') t_1^{Q-n-1} + \cdots + a_{Q-n}(t')) h(t), \quad t' = (t_1, \ldots, t_{n-1}), \]
where $a_1, \ldots, a_{Q-n}$ and $h$ are analytic functions in a neighborhood of 0, and $h(0) \neq 0$. (See [H].) Then
\begin{align*}
\int_{\sup \tilde{\psi}} |J^{(l)}(t)|^{-\alpha} \, dt &= 2^{-l(Q-n)\alpha} \int_{\sup \tilde{\psi}} |J(2^{-l} t)|^{-\alpha} \, dt \\
&\leq C 2^{-l(Q-n)\alpha} \int_{\frac{1}{2} \leq |t| \leq 2} 2^{l(n-Q)\alpha} |t_n^{Q-n} + 2^l a_1(t') t_n^{Q-n-1} + \cdots + 2^{l(Q-n)} a_{Q-n}(t')|^{-\alpha} \, dt \, dt' \\
&\leq CA_n \int_{1 \leq |t'| \leq 2} (1 + 2^l |a_1(t')| + \cdots + 2^{l(Q-n)} |a_{Q-n}(t')|)^{-\alpha} \, dt' \\
\end{align*}
which is bounded independent of $l$. The last inequality follows from [RS1] and the proof of the theorem is complete.

The general case where $\gamma$ is any connected analytic curve in $G$, follows by transference [CW, RS2] as is shown in the following remarks.

Remark. (vi) Assume now that $\gamma$ generates a proper subgroup of $G$. As in Remark (ii) we choose $\{e_1, \ldots, e_l\}$ that generate a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{g}$. Extend it to a basis $\{e_1, \ldots, e_l, \ldots, e_n\}$ of $\mathfrak{g}$ and define for $r > 0$, $D_r(x_1, \ldots, x_l, \ldots, x_n) = (r^{a_1} x_1, \ldots, r^{a_l} x_l, \ldots, r^{a_n} x_n)$ where $a_1, \ldots, a_l$ are determined by $\gamma$ and $a_{l+1}, \ldots, a_n$ are arbitrary integers. Assume that $D_r$ are automorphisms of $G$, let $\gamma_0$ be the homogeneous curve (w.r.t. $D_r$) associated to $\gamma$, and let $H$ be the subgroup of $G$ generated by $\gamma_0$.

Then $H$ is homogeneous with the dilations induced and $\gamma$ generates $H$. By transference, $Tf = f \ast \mu$ is bounded on $L^p(G)$.

Remark. (vii) Let $\gamma$ be an analytic curve that generates a connected and simply connected, nilpotent Lie Group $N$. Let $D_r$ be a group of dilation associated to
\( \gamma \), which are not necessarily automorphisms of \( N \) and let \( \gamma_0 \) be as in Remark (ii). As in [RS2] let \( g \) be the step \( m \) free Lie algebra generated by \( n \), for \( m \) large enough and let \( G \) be the simply connected group with \( \mathcal{L}(G) = g \). Then there exists \( H \) a normal subgroup of \( G \) such that \( G/H \) is isomorphic to \( N \). Denote by \( \sigma : g \rightarrow n \) the quotient morphism. By construction of \( g \) the dilations \( D_n \) on \( n \) extend to automorphisms \( \tilde{D}_n \) of \( g \).

Let \( \tilde{\gamma} = \exp_n^{-1} \circ \gamma \), \( \tilde{\gamma} = \exp_G \circ \tilde{\gamma} \) and \( \tilde{\gamma}_0 = \exp_n^{-1} \circ \gamma_0 \), \( \tilde{\gamma}_0 = \exp_G \circ \gamma_0 \). Then \( \tilde{\gamma}_0 \) is a homogeneous curve w.r.t. \( \tilde{D}_n \) and the coordinates of \( \tilde{\gamma}_0 \) (resp. \( \tilde{\gamma} \)) with respect to \( \exp_G \) are those of \( \gamma_0 \) (resp. \( \gamma \)). Thus \( Tf = f \ast \mu \) is bounded on \( L^p(G) \), \( 1 < p < \infty \) and by transference on \( L^p(N) \), \( 1 < p < \infty \).

**Remark.** (viii) The maximal function along \( \gamma \),

\[
M_\gamma f(x) = \sup_{r \leq 1} \frac{1}{r} \left| \int_{0 \leq |t| \leq r} |f(x\gamma(t)^{-1})| \, dt \right|
\]

is bounded on \( L^p(G) \), \( 1 < p \leq \infty \). Also

\[
T^* f(x) = \sup_r \left| \int_{r \leq |t| \leq 1} f(x\gamma(t)^{-1})dt/t \right|
\]

is bounded on \( L^p(G) \), \( 1 < p < \infty \) and hence \( Tf(x) \) exists a.e. for all \( f \in L^p(G) \), \( 1 < p < \infty \).

To see this we first remark that if \( Hf = \sup_j |f \ast f_j| \) with \( f_j \) satisfying

(i) \( \|f_j\|_1 \leq C_1 \),

(ii) \( f |f_j(xy) - f_j(x)| \leq C_2 j^j |y|^{\epsilon} \forall \epsilon > 0 \),

(iii) \( \sup f_j \subset \{|x| \leq C2^{-j}\} \).

Then \( H \) is bounded on \( L^\infty \) and of weak type \((1,1)\) with constant \( C_1 C_\epsilon \). Indeed, we can assume that \( f \geq 0 \). For \( \lambda > 0 \) we decompose \( f = g + b \) where \( \|g\|_\infty \leq \lambda \), \( b = \sum_j b_j \), \( \sup b_j \subset B_j = B(a_j, r_j) \), \( \int b_j = 0 \), \( \int |b_j| \leq A\lambda |B_j^*| \), and \( \sum |B_j^*| \leq \frac{2}{\epsilon} \|f\|_1 \), \( B_j^* = B(a_j, 2r_j) \). Then it is enough to estimate \( \{|x| \leq \lambda \} \). But

\[
\{x | Hb(x) > \lambda \} \subset \left( \bigcup_j B_j^* \right) \cup \left\{ x \in \left( \bigcup_j B_j^* \right) | Hb(x) > \lambda \right\}
\]
\[
\left\{ x \in \left( \bigcup B_j^* \right) \mid Hb(x) > \lambda \right\} \\
\leq \frac{1}{\lambda} \int_{\bigcup B_j^*} Hb(x) \leq \frac{1}{\lambda} \sum_i \int_{B_i^*} Hb_i(x) \\
= \frac{1}{\lambda} \sum_i \int_{B_i^*} \sup_j \left| \int_{B_i} b_j(y)f_j(y^{-1}x) \, dy \right| \, dx \\
= \frac{1}{\lambda} \sum_i \int_{B_i} \sup_j \left| \int_{B_i} b_j(y)(f_j(y^{-1}x) - f_j(a_i^{-1}x)) \, dy \right| \, dx \\
\leq \frac{1}{\lambda} \sum_i \int_{B_i} |b_i(y)| \sum_j \int_{c_B^*} |f_j(y^{-1}x) - f_j(a_i^{-1}x)| \, dx \, dy \\
\leq \frac{1}{\lambda} \sum_i \int_{B_i} |b_i(y)| \sum_j \int_{|z| > 2|y|} |f_j(y^{-1}z) - f_j(z)| \, dz \\
\leq C_\epsilon \cdot \frac{1}{\lambda} \sum_i \int_{B_i} |b_i(y)| \, dy \quad \text{by (ii) and (iii), } \hat{y} = y^{-1}a_i
\]

Now
\[
M_jf(x) \leq C \sup (f * |\mu_j|)(x)
\]
and
\[
\sup_j (f * |\mu_j|) \leq \sum_{k \geq 1} \sup_j |f * \eta_{k+j} * |\mu_j|| + \sup_j (f * \varphi_j * |\mu_j|) = \sum_k \tilde{M}_k + \tilde{N}.
\]

By the above \( \tilde{M}_k \) is bounded on \( L^\infty \) and for each \( \epsilon > 0 \) there exists a constant \( C_\epsilon \) such that \( \tilde{M}_k \) is of weak type \( (1,1) \) with constant \( C_\epsilon 2^{k\epsilon} \). The same holds for \( \tilde{N} \). We also have that \( \|\tilde{M}_k\|_{2,2} \leq C_2^{-\sigma k} \) for some \( \sigma > 0 \), by using the technique of the square functions (see [Ch]). Indeed we have to check that the operators \( S_k f(x) = \sum_j a_j f * \eta_{k+j} * |\mu_j|, a_j = \pm 1 \forall j \), have \( L^2 \) boundedness \( C_2^{-\sigma k} \) but this follows as in the proof of the theorem since \( n \)-times \( |\mu_j| * \cdots * |\mu_j| \) is absolutely continuous.

The boundedness of \( T^* \) on \( L^p, 1 < p < \infty \), reduces to prove that the operators
\[
T_1 f(x) = \sup_{i \geq 0} \left| f * \sum_{j \geq i} \mu_j * \varphi_i \right|
\]
and
\[
T_2 f(x) = \sup_i \left| \sum_{j=0}^i f * \mu_j * (\delta - \varphi_i) \right|
\]
are bounded on \( L^p \) for \( 1 < p < \infty \) (see [Ch]). \( T_1 f(x) \) is bounded pointwise by the maximal operator on \( G \) since the functions \( \psi_i = \sum_{j > i} \mu_j * \varphi_i \) satisfy \( \sup \psi_i \subset \{ x \mid |x| \leq C2^{-i} \} \) and \( |\psi_i(x)| \leq C2^i \sigma \).

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Now \( T_2 f(x) \leq \sum_k \sup_j |f \ast \mu_j \ast (\delta - \varphi_{k+j})| = \sum_k \widetilde{M}_k \) and \( \widetilde{M}_k f \leq M_k f + \sup_j |f \ast \mu_j \ast \varphi_{k+j}|. \) Since \( \sup_j |f \ast \mu_j \ast \varphi_{k+j}| \) is bounded on \( L^\infty \) and of weak type \((1,1)\), it is bounded on \( L^p \) for \( 1 < p < \infty \) and so is \( \widetilde{M}_k \). To see that \( \|\widetilde{M}_k\|_{2,2} \leq C 2^{-\sigma k} \), for some \( C, \sigma > 0 \), we can argue as in the theorem since \( n\)-times \( \mu_j \ast \cdots \ast \mu_j \) is absolutely continuous with density \( \rho_j \) satisfying the equation \( \int |\rho_j(xy) - \rho_j(x)| \leq C2^{jC}|y|^\epsilon \) and \( (\delta - \varphi_{k+j}) \) is a measure with bounded \( L^1 \)-norm, \( \text{supp} (\delta - \varphi_{k+j}) \subset \{ x ||x| \leq C2^{-j(k+j)} \} \) and it has mean value zero.

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References


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