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# NO CONTINUUM IN $E^2$ HAS THE TMP. I. ARCS AND SPHERES

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ABSTRACT. A subset $X$ of the Euclidean plane $E^2$ is said to have the triple midset property (TMP) if, for each pair of points $x$ and $y$ of $X$, the perpendicular bisector of the segment joining $x$ and $y$ intersects $X$ at exactly three points. In this paper it is proved that no arc or simple closed curve in $E^2$ can have the TMP. In a subsequent paper these results are used to prove that no planar continuum can have the TMP.

## 1. INTRODUCTION

Let $(X, \rho)$ be a metric space, and let $x$ and $y$ be two points of $X$. The *midset* $M(x, y)$ of $x$ and $y$ is the set of all points $m$ of $X$ such that $\rho(x, m) = \rho(y, m)$. Midsets have also been called bisectors [3] or equidistant sets [9, 10]. If there exists an integer $n$ such that, for each pair $x$, $y$ of distinct points of $X$, $M(x, y)$ consists of $n$ points, then $X$ is said to have the *n-Midset Property* (*n-MP*). When $n = 1$, this property has been called the *Unique Midset Property* (*UMP*) [2], for $n = 2$ it is called the *Double Midset Property* (*DMP*) [7], and for $n = 3$ it is also known as the *Triple Midset Property* (*TMP*) [7]. This paper, the first of two on the TMP, gives a proof that no arc or simple closed curve in the plane can have the TMP. The second paper uses these results to establish the more general theorem that no planar continuum has the TMP.

My interest in midset properties began with Berard's result [2] that a connected metric space with the UMP is homeomorphic to a subset of the real line. Generalizing, one might guess that each connected metric space with the DMP could be topologically embedded in the Euclidean plane $E^2$. In the compact case this would follow from the Double Midset Conjecture [7] that a *continuum* (a compact, connected, metric space containing more than one point) with the

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DMP must be a simple closed curve (a homeomorphic image of a circle). That conjecture remains open; however, I proved [5] that the only planar continuum (a continuum lying in $E^2$) with the DMP is a simple closed curve. Generalizing even more liberally, one might speculate that continua with the TMP lie in $E^3$; however, I doubt that continua with the TMP exist. Question 4 of [7] asks if a continuum can have the TMP. Unable to resolve the general question, I focused on continua with the TMP that might lie in $E^2$. Conjectures relating to midset properties are given at the end of the paper.

For a set $X$ in $E^2$, these midset properties are weaker than requiring that every line intersect $X$ in a specified number of points. Mazurkiewicz [8] showed the existence of a subset $X$ of the plane such that every line meets $X$ in exactly two points. Such a set has been called a two-point set; every two-point set has the DMP but not conversely. Bagemihl and Erdös [1] proved a general intersection theorem from which the existence of a three-point subset of $E^2$ followed. Although a three-point set cannot be a continuum, such a set has the TMP. Larman [4] proved that no $F_\sigma$ subset of the plane can be either a two-point or a three-point set. Mauldin (unpublished) proved that a two-point, planar set must be totally disconnected.

Let $X$ be a continuum in $E^2$, and let $a$ and $b$ be two points of $X$. It is useful, and in this paper essential, to distinguish between the *bisector* $B(a, b)$ of $a$ and $b$, the line perpendicularly bisecting the segment $[a, b]$ in $E^2$, and the *midset* $M(a, b)$ defined as $B(a, b) \cap X$. A *side* of a line $L$ is a component of $E^2 - L$, and the side of $L$ containing a point $p$ is called the *p-side* of $L$. The standard Euclidean metric $\rho$ is used for $E^2$.

An arc or a simple closed curve $A$ is said to *cross a line* $L$ in $E^2$ at a point $m$ if there are subarcs $A'$ and $A''$ of $A$ such that $A' \cap A'' = \{m\}$ and $A'$ and $A''$ lie on opposite sides of $L$. The arc $A$ is said to *bounce off* $L$ at a point $m$ if there is a subarc $A'$ of $A$ such that $m$ lies in the interior of $A'$, $A' \cap L = \{m\}$, and $A' - \{m\}$ lies in one side of $L$. Also, $A$ is said to *bounce off a bisector* $B(a, b)$ *at* $m$ *to the a-side of* $B(a, b)$ if $A' - \{m\}$ lies on the $a$-side of $B(a, b)$. An arc $A$ is said to *hang to the side* $S$ *of a line* $L$ *at a point* $v \in L$ if $v$ is an endpoint of $A$ and there exists a neighborhood $V$ of $v$ such that $(A - \{v\}) \cap V \subset S$.

## 2. No simple closed curve in the plane has the TMP

Used frequently in the proof of Theorem 2.1, the following lemma establishes a fundamental geometric principle for later reference.

**Lemma 2.1.** *If* $X \subset E^2$, $X$ *has the TMP,* $C$ *is a circle centered at* $t$, $U$ *is a component of* $E^2 - C$, $a$ *and* $b$ *are points of* $C \cap X$, *and* $X$ *contains three disjoint arcs, two that cross* $B(a, b)$ *and one, say* $A$, *that bounces off* $B(a, b)$ *at a point* $t$, *then* $a$ *and* $b$ *cannot both be limit points of* $U \cap X$. *If, in addition,* $A$ *bounces off* $B(a, B)$ *to the* $a$*-side of* $B(a, b)$ *at* $t$, *then* $a$ *cannot be a limit point of* $X \cap \operatorname{Ext} C$ *and* $b$ *cannot be a limit point of* $X \cap \operatorname{Int} C$.

*Proof of Lemma* 2.1. Suppose $a$ is a limit point of $X \cap \operatorname{Ext} C$, and let $A$, $P$, and $Q$ be the hypothesized disjoint arcs in $X$ such that $A$ bounces off $B(a, b)$ to the $a$-side $B(a, b)$ at $t$ and both $P$ and $Q$ cross $B(a, b)$. Choose a point $a'$ in $X \cap \operatorname{Ext} C$ such that $B(a', b)$ intersects both $P$ and $Q$. Since $a' \in \operatorname{Ext} C$, $t$ must lie in the $b$-side of $B(a', b)$ This ensures that $a'$ can be chosen close enough to $a$ that $B(a', b)$ intersects $A$ twice near the bounce point $t$. But this contradicts the TMP. A similar contradiction is exhibited if $b$ is a limit point of $X \cap \operatorname{Int} C$ by choosing a point $b'$ in $X \cap \operatorname{Int} C$ such that $B(a, b')$ contains four points of $X$. This proves the second part of Lemma 2.1, and the first part is similar.

**Theorem 2.1.** *No simple closed curve in the Euclidean plane has the triple midset property.*

*Proof of Theorem* 2.1. Suppose $X$ is a simple closed curve in the Euclidean plane $E^2$ such that $M(x, y)$ consists of three points for each two distinct points $x$ and $y$ of $X$. For each two points $x$ and $y$ of $X$, $X$ must cross $B(x, y)$ at two points and $X$ must bounce off $B(x, y)$ at the third point of $X \cap B(x, y)$ because $X$ is a simple closed curve and $B(x, y)$ separates $x$ from $y$. This fact will be used without reference in the sequel.

Let $a$ and $b$ be two points of $X$ such that the diameter of $X$ is $\rho(a, b)$. This means $X$ lies in the intersection of two disks $D$ and $E$, centered at $a$ and $b$, respectively, each with radius $\rho(a, b)$. Let $A$ and $A'$ be the two arcs in $X$ whose intersection is $\{a, b\}$ such that $A$ bounces off $B(a, b)$ at a point $t$ to the $a$-side of $B(a, b)$, and let $C$ be the circle centered at $t$ with radius $\rho(a, t)$. Impose a rectangular coordinate system such that $a$ and $b$ lie on the $x$-axis, $B(a, b)$ is the $y$-axis, $t = (0, \delta)$. Without loss of generality, assume $\delta \geq 0$. Let $U$ and $V$ be the closures of the upper (positive $y$-coordinate) and lower components, respectively, of $(D \cap E) - (C \cup \operatorname{Int} C)$, and note that $V \neq \phi$ because $\delta \geq 0$. By the second part of Lemma 2.1, $a$ and $b$ cannot be limit points of $X \cap \operatorname{Ext} C$ and $X \cap \operatorname{Int} C$, respectively, so if $\delta > 0$ it follows that $X$ bounces off the $x$-axis into $V$ at $b$. This could also happen if $\delta = 0$.

Case 1. $X$ bounces off the $x$-axis at $b$. With no loss of generality, assume $X$ bounces off the $x$-axis into $V$ at $b$. Let $G$ and $G'$ be the components of $A \cap V$ and $A' \cap V$ containing $b$, respectively, and let $\{d, b\}$ and $\{d', b\}$ be the sets of endpoints of $G$ and $G'$, respectively. In the circular arc $C \cap Bd\, V$, suppose $d'$ lies between $d$ and $b$. Then, since $t \in A$ and $G'$ crosses $B(b, d')$, it is clear that $A'$ must cross $B(b, d')$ at least three times. But the order of the points on $C \cap Bd\, V$ implies that $G$ also intersects $B(b, d')$. This contradicts the TMP, so $d$ must lie between $d'$ and $b$ on $C \cap Bd\, V$.

The order of $d'$, $d$, and $b$ on $C \cap Bd\, V$ implies that there exists a component $G^*$ of $G' \cap \operatorname{Ext} C$ with endpoints $b$ and $d^*$ such that $d$ lies between $d^*$ and $b$ on $C \cap Bd\, V$. Since $d^*$ and $b$ are each limit points of $X \cap \operatorname{Ext} C$, it follows from the first part of Lemma 2.1 that $A$ must cross $B(d^*, b)$ at the point $t$. Then $d^* \neq a$, and it follows that $B(d^*, b) - \{t\} \subset \operatorname{Int} S$, where $S$ is the double sector at $t$ such that $B\,dS = B(a, b) \cup B(d, b)$ and $\{a, b\} \cap S = \emptyset$. Since $G$ and $G'$ each cross $B(d, b)$ and $t \notin G \cup G'$, the simple closed curve $X$ must bounce off $B(d, b)$ at $t$. Because $X$ bounces off $B(a, b)$ at $t$ to the $a$-side of $B(a, b)$ and $X$ crosses $B(d^*, b)$ at $t$, $X$ must bounce off $B(d, b)$ at $t$ to the $b$-side of $B(d, b)$. Since $a$ and $d$ lie on the $d$-side of $B(d, b)$, the subarc $[a, d]$ of $A$ must cross $B(d, b)$ at two distinct points. Then $B(d, b) \cap A$ contains three points, and $B(d, b) \cap A'$ contains at least one point. But this contradicts the TMP, so $X$ cannot bounce off the $x$-axis at $b$.

Case 2. $X$ crosses the $x$-axis at $b$. In this case $\delta = 0$. With no loss of generality, assume $A'$ and $A$ hang off the $x$-axis at $b$ into $V$ and $U$, respectively. Suppose there is no arc in $A \cap C \cap U$ having $b$ as an endpoint. Then there exists an arc $F$ in $A$ joining $b$ to a point $b'$ such that $F - \{b, b'\} \subset C \cup \operatorname{Ext} C$, $b' \in C$, and both $b$ and $b'$ are limit points of $A \cap \operatorname{Ext} C$. Since $a$ is not a limit point of $X \cap \operatorname{Ext} C$, $a \neq b'$. From Lemma 2.1, $A$ must cross $B(b, b')$ at $t$. But $F$ also crosses $B(b, b')$, and, since $B(b, b')$ separates $a$ from $b$, $A$ must cross $B(b, b')$ a third time. However, $B(b, b')$ also intersects $A'$, which contradicts the TMP.

From the previous paragraph, there exists an arc $K$ in $A \cap C \cap U$ such that $b \in K$. Let $b'$ be the other endpoint of $K$. Then, for every two points $x$ and $y$ in $K$, $X$ must bounce off $B(x, y)$ at $t$. To see this, suppose $x$ and $y$ exist in $K$ such that $X$ crosses $B(x, y)$ at $t$. Then the simple closed curve $X$ must bounce off $B(x, y)$ at some other point $t'$. For $x'$ and $y'$ between $x$ and $y$ in $K$ and sufficiently close to $x$ and $y$, respectively, one of $B(x, y')$ and $B(x', y)$ must contain four points of $X$—two near $t'$, $t$, and a fourth point in $K$. This contradiction to the TMP shows that, at $t$, $X$ bounces off the entire double sector $S$ of lines passing through $t$ and intersecting $K$. Furthermore, to avoid crossing lines in $S$ three times at points other than $t$, $A$ must bounce off $S$ at $t$ above the $x$-axis. Label the four closed quadrants of $E^2$ in the usual counterclockwise manner as $Q_i$, $i \in \{1, 2, 3, 4\}$, and note that, for similar reasons, $A$ bounces off every line in $Q_1 \cup Q_3$ to its upper or $Q_2$-side at $t$.

Suppose there exists a point $v \in V \cap C$ such that $v \neq a$ and $v$ is a limit point of $X \cap \operatorname{Int} C$. Since $B(a, v) \subset Q_1 \cup Q_3$, $A$ bounces off $B(a, v)$ at $t$ to the $a$-side of $B(a, v)$. This contradicts Lemma 2.1, and it follows that no point of $C \cap X \cap V - \{a\}$ can be a limit point of $X \cap \operatorname{Int} C$.

Suppose $V \cap C \subset A'$. Let $E$ be the component of $A - C$ containing $t$, and let $r$ and $s$ be the endpoints of $E$. Since $r$ and $s$ are each limit points of $X \cap \operatorname{Int} C$, it follows from Lemma 2.1 that $A$ cannot bounce off $B(r, s)$ at $t$. Let $t''$ be the point of $X$ where $X$ bounces off $B(r, s)$. The supposition that $V \cap C \subset A'$ ensures that $B(r, s)$ is not the $x$-axis, so, by the supposition, $B(r, s)$ must intersect $V \cap C$ at a point $e$ between $a$ and $b$. This means there exist points $r'$ and $s'$ in $V \cap C \cap A'$ such that $B(r, s) = B(r', s')$. Choose a point $z$ between $r'$ and $s'$ in $A'$ close enough to either $r'$ or $s'$, depending on the side of $B(r', s')$ to which $X$ bounces at $t''$, that either $B(z, s')$ or

$B(r', z)$, say $B(r', z)$, intersects $X$ twice near $t''$. Then $B(r', z)$ contains $t$, and $B(r', z)$ must intersect $A' \cap C$ at a point $f$ near $e$. Since $f \neq t''$, $M(r', z)$ contains four points. This contradicts the TMP, so the lower half of $C$ cannot lie in $A'$.

From the previous paragraph, the components $H$ and $H'$ of $A' \cap C \cap V$ containing $a$ and $b$, respectively, are disjoint. Let $\{a, p\}$ and $\{b, q\}$ be the set of endpoints of $H$ and $H'$, respectively, with the understanding that $a = p$ if $H = \{a\}$ and $b = q$ if $H' = \{b\}$. Since no point of $V \cap C - \{a\}$ is a limit point of $X \cap \operatorname{Int} C$ (see the third paragraph of the proof of this Case 2), $q$ is not a limit point of $A' \cap \operatorname{Ext} C$. Suppose $p = a$. Then $p$ is a limit point of $A' \cap \operatorname{Int} C$, and there must exist an arc $G$ in $A'$ from $a$ to a point $g$ in $C \cap V - \{b\}$ such that $G - \{a, g\} \subset \operatorname{Int} C$. This makes $g$ a limit point of $A' \cap \operatorname{Int} C$, which contradicts the fact that no point of $C \cap V - \{a\}$ can be a limit point of $X \cap \operatorname{Int} C$. Therefore, $p \neq a$, and it follows that $p$ and $q$ are both limit points of $X \cap \operatorname{Ext} C$. By Lemma 2.1, $A$ must cross $B(p, q)$ at $t$, so there must exist a point $t'$ such that $X$ bounces off $B(p, q)$ at $t'$ and $t \neq t'$. Select points $p' \in H - \{p\}$ and $q' \in H' \cup K - \{q\}$ close to $p$ and $q$, respectively, such that one of the two bisectors $B(p', q)$ and $B(p, q')$ intersects $X$ at two points near the bounce point $t'$. This bisector will contain $t$ and a fourth point of $X$ near to where $A'$ crosses $B(p, q)$. This contradiction to the TMP shows that the arc $K$ does not exist, and Theorem 2.1 follows.

## 3. No planar arc has the TMP

If $I$ is a straight line segment, then $L(I)$ denotes the set of all lines intersecting $I$ that are perpendicular to $I$. If $X \subset E^2$ and every line in $L(I)$ is a bisector for two points of $X$, then $I$ is called an *interval of bisectors* for $X$. In several places in the proof of Theorem 3.1, a line $L$ is identified and $\psi$ is used to denote the reflection of $E^2$ in the line $L$. When the line $L$ appears in the proof of Theorem 3.1 it will be clear that lines parallel to $L$ and in a small neighborhood of $L$ form an interval of bisectors that contradict Lemma 3.1 below.

**Lemma 3.1.** *If $X$ is an arc in $E^2$, $X$ has the TMP, $a$ and $b$ are the endpoints of $X$, and $I$ is an interval of bisectors for $X$, then there exists a line in $L(I)$ that separates $a$ from $b$.*

*Proof of Lemma* 3.1. Suppose no line in $L(I)$ separates $a$ from $b$, and select $L \in L(I)$ such that $L$ intersects $\operatorname{Int} I$. Since $L$ is a bisector for $X$ and $X$ has the TMP, $L \cap X$ consists of three points. Because $L$ does not separate $a$ from $b$, $X$ must bounce off $L$ at some point $t$, say to the $a$-side of $L$, and $X$ must cross $L$ at the other two points of $X \cap L$. However, this contradicts the TMP because there must exist a line $L' \in L(I)$ near $L$ and on the $a$-side of $L$ such that $L' \cap X$ contains four points—two near $t$ and one near each point where $X$ crosses $L$. This establishes the lemma.

**Theorem 3.1.** *No planar arc has the triple midset property.*

*Proof of Theorem* 3.1. Suppose $X$ is an arc in $E^2$ such that $X$ has the TMP, and let $a$ and $b$ be the endpoints of $X$. The interval notation will be used on $X$ in such a way as to be consistent with $X = [a, b]$. Note that the midset $M(a, b)$ consists of three points $c$, $d$, $e$, and either exactly one or all three of them are points where $X$ crosses the bisector $B(a, b)$. If all three are crossing points for $X$, assertion (1) below gives the contradiction. Assertions (2) and (3) produce the contradiction for the other case, and Lemma 3.1 is used throughout to make the contradiction explicit.

Impose a rectangular coordinate system such that $B(a, b)$ is the $y$-axis and the line through $a$ and $b$ is the $x$-axis. If $p$ and $q$ are points on the $y$-axis, it is convenient to use the notation "$p < q$" to mean that the $y$-coordinate of $p$ is smaller than the $y$-coordinate of $q$. Also, the direction of the positive $y$-axis is called the upward direction. If $L$ is a line then $\pi_L \colon E^2 \to L$ is the orthogonal projection, while $\pi$ denotes this projection onto the $x$-axis.

(1) There cannot be three points of $B(a, b)$ where $X$ crosses $B(a, b)$.

*Proof of* (1). Suppose $X$ crosses $B(a, b)$ at each of the three points $c$, $d$, and $e$ of $X \cap B(a, b)$. Assume the three points are named to have the order $acdeb$ on $X$, and note that the subarcs $[a, c)$ and $(d, e)$ of $X$ lie on the $a$-side of $B(a, b)$ while $(c, d) \cup (e, b]$ lies on the $b$-side. By renaming the points $a$, $b$, $c$, $d$, or $e$ or by choosing the coordinate system differently, if necessary, it may be assumed that two of the three points $c$, $d$, and $e$ have nonnegative $y$-coordinates, that $e < c$, and $0 \leq c$, where 0 denotes the origin. This leaves only three possible orders of the three points on $B(a, b)$. Let $F$ be the disk whose boundary is the union of the arc $[d, e]$ in $X$ and the segment of $B(a, b)$ joining $d$ and $e$.

Suppose $e < d < c$ on $B(a, b)$, and choose a line $L$ parallel to $B(c, d)$, and slightly below it, such that $e < \psi(c) < d$. Note that $[a, c] \cap F = \varnothing$. Since $d \geq 0$, $L$ can be chosen above the $x$-axis, which means $L$ intersects $(a, c)$ at some point $p$. Then $\psi(p) = p \notin F$, so $\psi((p, c))$ must intersect $(d, e)$ at some point $x$. From this, $L = B(x, \psi^{-1}(x))$, and the same argument shows lines near $L$ and parallel to it form an interval of bisectors all lying above the $x$-axis. However, this contradicts Lemma 3.1.

Suppose $d < e < c$ on $B(a, b)$, and choose $L$ parallel to $B(c, e)$ and slightly below it such that $d < \psi(c) < e$ and $L$ does not intersect the $x$-axis. Since $0 \leq e$, $L$ separates $a$ from $c$, so $L$ intersects $(a, c)$ at some point $q$. Then $\psi((q, c))$ intersects $(d, e)$ at a point $x$, and $L = B(x, \psi^{-1}(x))$. A contradiction to Lemma 3.1 follows.

The only remaining case is where $e < c < d$, $0 < d$, and $0 \leq c$. Choose $L$ parallel to $B(c, e)$, and slightly below it, such that $\psi(c) < e$. Suppose $B(c, e)$ coincides with the $x$-axis. In this case $a \in \operatorname{Int} F$, and $L$ may be chosen so that the vertical segment from $a$ to $\psi(a)$ lies in $\operatorname{Int} F$. Since $\psi(a) \in \operatorname{Int} F$

and $\psi(c) \notin F$, $\psi((a, c))$ must intersect $(d, e)$ at a point $x$, and $L = B(x, \psi^{-1}(x))$. This leads to a contradiction of Lemma 3.1, so $B(c, e)$ is not the $x$-axis. Suppose $B(c, e)$ lies above the $x$-axis, and choose $L$ as before, but also above the $x$-axis. Then $[a, c)$ must intersect $L$ at a point $z$, and $\psi((z, c))$ must contain a point $x$ of $(d, e)$ since $\psi(z) = z \in \operatorname{Int} F$ and $\psi(c) \notin F$. Again Lemma 3.1 is contradicted since this leads to an interval of bisectors lying above the $x$-axis. Therefore, $B(c, e)$ lies below the $x$-axis. Forget the line $L$, and choose a horizontal line $T$ slightly above $B(c, e)$ such that $T$ also lies below the $x$-axis. If $\psi_T$ is the reflection of $E^2$ in $T$, then select $T$ so that $c < \psi_T(e) < d$. It follows that $[e, b]$ intersects $T$ at a point $z$, and $\psi_T((e, z))$ must intersect $[c, d]$ at a point $x$. This shows that $T$ and lines parallel to and near $T$ are all bisectors, so an interval of bisectors exists contrary to Lemma 3.1, and (1) follows.

(2) The arc $X$ cannot bounce off $B(a, b)$ to the same side of $B(a, b)$ at two of the points in $\{c, d, e\}$.

*Proof of* (2). Suppose $X$ bounces off $B(a, b)$ to the side $S$ of $B(a, b)$ at each of the points $c$ and $e$. No loss of generality occurs by assuming $S$ is the $a$-side of $B(a, b)$, and, since the point $d$, where $X$ crosses $B(a, b)$, cannot lie between $c$ and $e$ on $X$, there is no loss in assuming the order $acedb$ on $X$. In the first of two cases, suppose $d$ lies either above or below both $c$ and $e$ on $B(a, b)$, let $p$ denote the point of $\{c, e\}$ closer to $d$, and let $q = \{c, e\} - \{p\}$. Choose a line $L$ parallel to $B(d, p)$, slightly below it if $p < d$ and slightly above $B(d, p)$ if $d < p$, but close enough to $B(d, p)$ that $\psi(d)$ lies between $p$ and $q$ on $B(a, b)$. Clearly $L$ can also be chosen to miss the $x$-axis. Since $L$ intersects $(e, d)$ at some point $z$, $\psi((z, d))$ must intersect $(c, e)$ at a point $x$. Then $L = B(x, \psi^{-1}(x))$, which leads to a contradiction with Lemma 3.1. In the last case, suppose $d$ lies between $c$ and $e$, and select $L$ parallel to $B(d, c)$ in such a way that $\psi(c)$ lies between $d$ and $e$. Since $L$ must intersect $(c, e)$ at a point $z$, it follows that $\psi((z, c))$ intersects $(d, e)$ at a point $x$. But the bisector $L$ can be chosen to miss the $x$-axis, so an interval of bisectors is produced near $L$ in which each line fails to separate $a$ from $b$. This contradiction to Lemma 3.1 establishes (2).

(3) The arc $X$ cannot bounce off $B(a, b)$ to opposite sides of $B(a, b)$ at two of the points in $\{c, d, e\}$.

*Proof of* (3). Suppose (3) is false, and let $d$ denote the point where $X$ crosses $B(a, b)$. Since $d$ must lie between $c$ and $e$ on $X$, assume with no loss of generality, that the order $acdeb$ exists on $X$ and that $0 \le d$. Let $m$ denote the point of $\{c, d, e\}$ that is between the other two on $B(a, b)$. The proof breaks into cases depending upon the identity of $m$. In each case lines $M$ and $L$ are chosen such that $M$ is nearly vertical, very close to $B(a, b)$, and $m \in M$, while $L$ is perpendicular to $M$. As before, $L$ turns out to be a bisector for $X$ and forms the basis for an interval of bisectors in contradiction to Lemma 3.1. Let $C$, $D$, and $E$ be disjoint open balls centered at $c$, $d$, and $e$, respectively, whose diameters are very small compared to $\rho(a, b)$ and to the pairwise distances among $c$, $d$, and $e$, such that each ball intersects $X$ in an open arc. In every case below, $M$ is chosen to intersect each of $C$, $D$, and $E$ near their centers.

Suppose $m = d$ and, further, that $e < d < c$ on $B(a, b)$. Because $0 \leq d$, $B(c, d)$ lies above the $x$-axis. Choose $M$ through $d$ close enough to $B(a, b)$ that $M$ intersects both $(a, c)$ and $(c, d)$ in $C$, and let $X'$ be an arc from $a$ to $b$, obtained by pushing $\operatorname{Int} X$ slightly to the appropriate side, such that $X'$ does not intersect $B(a, b)$ in $C$. Then the simple closed curve $X \cup X'$ bounds a disk $H$ such that $H \cap C - \{c\}$ lies on the $a$-side of $B(a, b)$. Let $K$ be the closure of a component of $M \cap \operatorname{Int} H$ such that $K$ intersects each of $(a, c)$ and $(c, d)$, and $K \subset C$. Let $f$ be the point of $K$ closest to $d$, and note that $f \neq d$. Choose a line $L$ parallel to $B(d, f)$, slightly above it, but near enough to $B(d, f)$ that $\psi(d) \in \operatorname{Int} K$ and $L$ fails to separate $a$ from $b$. Let $G'$ be the closure of the component of $H - L$ containing $K$, and let $G = G' \cup \psi(G')$. Since $\psi(d) \in \operatorname{Int} K$, $d \in \operatorname{Int} G$. Because $\pi([a, d]) \cap \pi(b) = \varnothing$, and $BdG' - \{c\}$ lies on the $a$-side of $B(a, b)$, the lines $L$ and $M$ can be chosen such that $\pi_L(G) \cap \pi_L(b) = \varnothing$. This ensures that $BdG$ separates $b$ from $d$. Then $(d, b)$ must intersect $BdG$ at a point $x$, and $x$ must lie in $\psi((a, d))$. As before, $L$ is the bisector $B(x, \psi^{-1}(x))$, which contradicts Lemma 3.1.

In the case where $m = d$ and $c < d < e$, the above paragraph applies to yield a contradiction if the names of $a$ and $b$ are interchanged and the names of $c$ and $e$ are interchanged.

Suppose $m = c$, and, further, that $e < c < d$ on $B(a, b)$. Choose the line $M$ through $c$ such that $M$ intersects $(c, d)$ at a point $y \in D$, $M$ intersects both $(d, e)$ and $(e, b)$ in $E$, and $(y, d) \subset D$. Let $X'$ be an arc from $a$ to $b$, obtained by pushing $\operatorname{Int} X$ slightly to the appropriate side of $X$, such that $X'$ does not intersect $B(a, b)$ in $E$, and let $H$ be the disk bounded by $X \cup X'$. Then $H \cap E - \{e\}$ lies on the $b$-side of $B(a, b)$. Let $K$ be an arc in $M \cap H$ such that $\operatorname{Int} K \subset \operatorname{Int} H$, $K$ intersects both $(d, e)$ and $(e, b)$, and let $f$ be the point of $K$ closest to $y$. Clearly, $f \neq y$ because these points lie on opposite sides of $B(a, b)$. There are two subcases. Suppose $B(d, e)$ is not the $x$-axis. Then $M$ can be chosen so that $B(y, f)$ does not separate $a$ from $b$. In this case, choose $L$ parallel to $B(y, f)$, slightly below it, but close enough to $B(y, f)$ and to $B(d, e)$ that $L$ does not separate $a$ from $b$, that $\psi(y) \in \operatorname{Int} K$, and that $\psi(b)$ lies on the $b$-side of $B(a, b)$. Let $G'$ be the closure of the component of $H - L$ containing $K$, and let $G = G' \cup \psi(G')$. Then $y \in \operatorname{Int} G$ because $\psi(y) \in \operatorname{Int} K \subset \operatorname{Int} G'$. Since $\pi(a) \cap \pi([d, b]) = \varnothing$ and $BdG' - \{e\}$ lies on the $b$-side of $B(a, b)$, $M$ and $L$ can be chosen such that $\pi_L(a) \cap \pi_L(G) = \varnothing$. This means $a$ does not lie in the convex hull of $G$, so $(a, y)$ must intersect $BdG$ at a point $x$ on the $a$-side of $B(a, b)$. Then $x \in \psi((d, b))$, so $L = B(x, \psi^{-1}(x))$, a contradiction to Lemma 3.1. Suppose $B(d, e)$ is the $x$-axis. Then $e < 0$, $e < c < d$, and $B(c, e)$ lies below the

$x$-axis. Choose $M$ such that $B(c, f)$ does not separate $a$ from $b$, and let $L$ be a line parallel to and just below $B(c, f)$, close enough to $B(c, f)$ to ensure that $L$ does not separate $a$ from $b$ and that $\psi(c) \in \operatorname{Int} K$. If $G$ and $G'$ are chosen as above, then $c \in \operatorname{Int} G$. With $M$ close enough to $B(a, b)$ that $\pi_L(a) \cap \pi_L(G) = \varnothing$, an arc in $\psi((d, b))$ separates $a$ from $c$ on the $a$-side of $B(a, b)$, so there must exist a point $x$ in its intersection with $(a, c)$. Then $L = B(x, \psi^{-1}(x))$, again contradicting Lemma 3.1.

The case where $m = c$ and $d < c < e$ is similar to the previous paragraph, except it is easier because, from $0 \leq d$, $B(d, e)$ can never coincide with the $x$-axis.

The final case is where $m = e$. But simple name changing shows this to be the same case as when $m = c$. Therefore, (3) follows, and so does Theorem 3.1.

**Theorem 3.2.** *If $X$ is a continuum in $E^2$ such that $X$ has the $n$-MP for some integer $n$, then:*

(1) *if $n = 1$, $X$ is an arc,*

(2) *if $n = 2$, $X$ is a simple closed curve, and*

(3) *if $n = 3$, $X$ cannot be an arc or a simple closed curve.*

*Proof.* Part (3) follows from Theorems 2.1 and 3.1, part (2) follows from Theorem 3 of [5], and part (1) is easily proven. Theorem 3.2 follows.

**Conjecture 3.1.** *No continuum has the $n$-MP for $n > 2$.*

**Conjecture 3.2.** *No planar continuum has the $n$-MP for $n > 3$.*

**Conjecture 3.3.** *No connected metric space containing more than one point has the $n$-MP for $n > 2$.*

**Conjecture 3.4.** *Among connected, nondegenerate metric spaces, only the simple closed curve has the DMP.*

A special case of Conjecture 3.1, [7, Question (4)] asks if a continuum can have the triple midset property. All four questions in [7, p. 1005] remain unresolved. Conjecture 3.4 was stated as a question in [3].

**Conjecture 3.5.** *No continuum (or nondegenerate connected set) in Euclidean $p$-space $E^p$ can have the $n$-MP for $n > 2$.*

Conjecture 3.5 is true for the case $n = 3$ and $p = 2$, see [6]. From the main result of [6], Theorem 3.2, part (3), can be strengthened to say that no planar continuum has the 3-MP.

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