A SPREAD RELATION FOR ENTIRE FUNCTIONS
WITH NEGATIVE ZEROS

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Abstract. Let $g$ be a canonical product having only real negative zeros and
nonintegral order $\lambda$, and let $\phi$ be the set function defined by $2\pi\phi(E) = \int_E \pi \csc \pi \lambda \cos \theta \, d\theta$. It is shown that if $E(r)$ is the set of values of $\theta \in (-\pi, \pi]$ where $|g(re^{i\theta})| \geq 1$, $r_n$ is a sequence of Polya peaks of $g$ and $\delta$ is
the deficiency of the value zero of $g$ then $\liminf \phi(E(r_n)) \geq 2(1 - \delta)^{-1}$. This
inequality leads to a sharp spread relation for $g$.

1. Introduction

Let

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp \left(\sum_{j=1}^{q} \frac{1}{j} \left(\frac{z}{a_n}\right)^{j}\right)$$

denote a Weierstrass canonical product of genus $q \geq 1$, having only negative
zeros $a_n$. Denote by

$$T(r, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |g(re^{i\theta})| \, d\theta$$

the Nevanlinna characteristic of $g$, and denote by

$$\mu = \liminf_{r \to \infty} \frac{\log T(r, g)}{\log r}$$

the lower order of $g$.

Let

$$E(r) = \{\theta \in [-\pi, \pi]: \log |g(re^{i\theta})| \geq 0\},$$

and denote by $|E(r)|$ the measure of the Lebesgue measurable set $E(r)$. Then it
is a consequence of the spread relation for meromorphic functions conjectured
by Edrei and proved by Baernstein [3] that

$$\liminf_{m \to \infty} |E(r_m)| \geq \frac{\pi}{\mu},$$
where $r_m$ is a sequence of Polya peaks of order $\mu$ of $T(r, g)$. The exact meaning of these peaks will not be needed here; for their definition see [4]. In private communication, Professor Daniel Shea asked whether (5) can be improved for the functions $g$ defined by (1) and having only negative zeros. It is the purpose of this note to obtain a sharp form of (5) for such functions.

Recall that the deficiency of the value 0 of $g$ is defined by

$$\delta(0) = \delta(0, g) = 1 - \limsup_{r \to \infty} \frac{N(r, 1/g)}{T(r, g)},$$

where

$$N(r, 1/g) = \int_0^r t^{-1} n(t) dt$$

and $n(t)$ is the number of zeros of $g$ in the disk $|z| \leq t$. Our results may be stated as follows:

**Theorem 1.** Let $g$ be a Weierstrass canonical product of genus $q \geq 1$, having only negative zeros and assume that the lower order $\mu$ of $g$ is nonintegral. Let $\psi_{\mu}$ be the function defined on $(-\pi, \pi)$ by

$$\psi_{\mu}(\theta) = \pi \mu \csc(\pi \mu) \cos \theta,$$

and introduce the set function $\phi$ defined on Lebesgue measurable subsets of $[-\pi, \pi]$ by

$$\phi(E) = \frac{1}{\pi} \int_E \psi_{\mu}(\theta) d\theta.$$

If $r_m$ is a sequence of Polya peaks of order $\mu$ of $T(r, g)$ and $\delta(0)$ is the deficiency of the value 0 of $g$, then

$$\liminf_{m \to \infty} \phi(E(r_m)) \geq \frac{2}{1 - \delta(0)}$$

and

$$\liminf_{m \to \infty} |E(r_m)| \geq \frac{2(q + 1)}{\mu} \sin^{-1}\left\{ \frac{|\sin \pi \mu|}{(q + 1)(1 - \delta(0))} \right\}.$$

Furthermore, both inequalities are sharp.

If the lower order of $g$ satisfies $q < \mu \leq q + \frac{1}{2}$, then it may happen that one component of the set $E(r_m)$ extends all the way up to $\pi$. In this case we can obtain a sharper form of (11):

**Theorem 2.** Let $g$ be as in Theorem 1 and assume that

$$q < \mu \leq q + \frac{1}{2}.$$

If $\pi \in E(r_m)$ for infinitely many values of $m$, then

$$\limsup_{m \to \infty} |E(r_m)| \geq 2\pi(1 - q/\mu) + \frac{2q}{\mu} \sin^{-1}\left\{ \frac{|\sin \pi \mu|\delta(0)}{q(1 - \delta(0))} \right\}.$$

Furthermore, (13) is sharp.
For the case \( 0 < \mu < 1 \), the analogues of (10) and (11) continue to hold true [4] and require no geometric restriction on the zeros of \( g \). As for (13), the spread relation of Edrei gives the stronger result \( \liminf |E(r_m)| = 2\pi \).

An entire function having finite lower order \( \mu \) and having all its zeros on one ray necessarily [2] has finite order \( \lambda \) and satisfies \( \lambda \leq [\mu] + 1 \). Thus for the functions \( g \) defined by (1) we always have \( q \leq \mu \leq \lambda \leq q+1 \). The assumptions in Theorems 1 and 2 imply \( q < \mu < q + 1 \).

Our proofs depend on the following two results of Hellerstein and Williamson [5]:

**Theorem A.** Let \( g \) be as in the statement of Theorem 1 and let

\[
C(r) = \{ \theta \in [0, \pi]: \log |g(re^{i\theta})| \geq 0 \}.
\]

Then for each \( r > 0 \), there exist points \( \alpha_1 = \alpha_1(r), \ldots, \alpha_{q+1} = \alpha_{q+1}(r) \) satisfying

\[
\frac{2j-1}{2(q+1)} \pi < \alpha_j < \frac{2j-1}{2q} \pi \quad (j = 1, 2, \ldots, q)
\]

and

\[
\frac{2q+1}{2(q+1)} \pi < \alpha_{q+1} \leq \pi,
\]

such that

\[
C(r) = \bigcup_{j=1}^{(q+1)/2} [\alpha_{2j-1}, \alpha_{2j}] \quad \text{if } q \text{ is odd}
\]

and

\[
C(r) = \bigcup_{j=0}^{q/2} [\alpha_{2j}, \alpha_{2j+1}], \alpha_0 = 0 \quad \text{if } q \text{ is even}.
\]

**Theorem B.** Let \( g \) be as in Theorem 1 and let \( r_m \) be a sequence of Polya peaks of order \( \mu \) of \( T(r, g) \). If

\[
\sigma > \limsup_{r \to \infty} \frac{N(r, 1/g)}{T(r, g)},
\]

then

\[
1 \leq \sigma (1 + o(1)) \csc \pi \mu \left[ \sum_{j=0}^{[(q+1)/2]} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j}) + o(1) \right]
\]

as \( m \to \infty \), where \( \alpha_{q+2} = 0 \).

The inequality in (20) above is a restatement of the inequality (4.17) in [5, p. 249] where we have divided by \( T(r_m, g) \). The dependence on the Polya peaks occurs in \( \alpha_j = \alpha_j(r_m) \).
2. Proof of Theorem 1

The sets \( E(r) \) and \( C(r) \) defined in (4) and (14) respectively satisfy \( E(r) = C(r) \cup -C(r) \). This is because the function \( |g(re^{i\theta})| \) is an even function of \( \theta \), a fact which follows from (1) and the reality of the zeros \( a_n \). Recalling the definitions of \( \psi_\mu(\theta) \) and \( \phi \) we have

\[
\frac{1}{2} \phi(E(r_m)) = \frac{1}{\pi} \int_{C(r_m)} \psi_\mu(\theta) d\theta.
\]

If \( q \) is odd, then by (17) the integral in (2.1) equals

\[
\csc \pi \mu \sum_{j=1}^{(q+1)/2} (\sin \mu \alpha_{2j} - \sin \mu \alpha_{2j-1}) = \left| \csc \pi \mu \right| \sum_{j=0}^{[q+1)/2} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j}),
\]

where we have used \( \alpha_0 = \alpha_{q+2} = 0 \) and \( q < \mu < q+1 \) to write \( \csc \pi \mu = -\left| \csc \pi \mu \right| \). If \( q \) is even, then in view of (18) the integral in (2.1) equals

\[
\csc \pi \mu \sum_{j=0}^{q/2} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j}) = \left| \csc \pi \mu \right| \sum_{j=0}^{[q+1)/2} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j}).
\]

Using (20) and the above, we obtain

\[
2 \leq \sigma(1 + o(1))\phi(E(r_m)) + o(1) \quad \text{as} \ m \to \infty.
\]

Now (10) follows immediately from (2.2), (19), and (6).

We note here that (2.2) and (2.1) imply

\[
2 \leq \sigma(1 + o(1))\phi(E(r_m)) + o(1)
\]

\[
\leq \sigma(1 + o(1)) \times \frac{2}{\pi} \int_{0}^{\pi} \psi_\mu^+(\theta) d\theta + o(1) \quad \text{as} \ m \to \infty.
\]

The first and last inequalities imply that

\[
\liminf_{r \to \infty} \frac{T(r, g)}{N(r, 1/g)} \leq \frac{1}{\pi} \int_{0}^{\pi} \psi_\mu^+(\theta) d\theta,
\]

which is the well-known result of Hellerstein and Williamson [5]. Note that (2.3) may be used to obtain information about functions extremal for the inequality (2.4). Indeed, if equality holds in (2.4), then (2.3) implies that \( \phi(E(r_m)) \to \frac{1}{\pi} \int_{0}^{\pi} \psi_\mu^+(\theta) d\theta \) as \( m \to \infty \), and we may conclude from this that

\[
\alpha_j(r_m) \to \frac{(2j - 1)}{2\mu} \pi \quad \text{as} \ m \to \infty,
\]

for \( j = 1, \ldots, q+1 \) if \( q + \frac{1}{2} < \mu \leq q+1 \), and for \( j = 1, \ldots, q \) and \( \alpha_{q+1} \to \pi \) if \( q < \mu \leq q + \frac{1}{2} \).
Proof of (11). Suppose first that \( q \) is odd. Using the elementary identity \( \sin x - \sin y = 2 \cos \left( \frac{x+y}{2} \right) \sin \left( \frac{x-y}{2} \right) \), we obtain

\[
|\csc \pi \mu| \sum_{j=0}^{(q+1)/2} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j})
\]

\[
= |\csc \pi \mu| \sum_{j=1}^{q+1} (-1)^{j-1} \sin \mu \alpha_j
\]

(2.5)

\[
= 2 |\csc \pi \mu| \sum_{j=1}^{(q+1)/2} \sin \frac{1}{2} \mu (\alpha_{2j-1} - \alpha_{2j}) \cos \frac{1}{2} \mu (\alpha_{2j-1} + \alpha_{2j})
\]

\[
\leq 2 |\csc \pi \mu| \sum_{j=1}^{(q+1)/2} \sin \frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}).
\]

To justify the last inequality in (2.5), we must show that \( \frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}) \leq \pi \) for \( j = 1, 2, \ldots, (q+1)/2 \). By (15) and (16) we have \( \alpha_{q+1} - \alpha_q \leq \pi - \frac{2q-1}{2(q+1)} \pi = \frac{3\pi}{2q+1} < \frac{2\pi}{q} \) since \( q < \mu < q+1 \). Also, if \( q > 1 \), then for \( j = 1, \ldots, (q-1)/2 \) we have by (15)

\[
\alpha_{2j} - \alpha_{2j-1} \leq \frac{4j-1}{2q} \pi - \frac{4j-3}{2(q+1)} \pi = \frac{4q+2q-1}{2q(q+1)} \pi
\]

\[
\leq \frac{4q-3}{2q(q+1)} \pi < 2\pi/(q+1) < 2\pi/\mu.
\]

Thus in all cases \( \frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}) \leq \pi \) and the inequality in (2.5) is justified.

To continue, note that the function \( \sin x \) is concave on \([0, \pi]\) since \((\sin x)^{''} = -\sin x < 0\) for all \( x \in [0, \pi] \). Thus \( \sin(n^{-1} \sum_{j=1}^{n} x_j) \geq n^{-1} \sum_{j=1}^{n} \sin x_j \) for any \( n \) numbers \( x_j \in [0, \pi] \). Taking \( x_j = \frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}) \) where \( j = 1, \ldots, (q+1)/2 \), we obtain

(2.6)

\[
\sum_{j=1}^{(q+1)/2} \sin \frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}) \leq \frac{q+1}{2} \sin \left\{ \frac{2}{q+1} \sum_{j=1}^{(q+1)/2} \frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}) \right\}
\]

\[
= \frac{(q+1)}{2} \sin \left\{ \frac{\mu}{q+1} |C(r_m)| \right\},
\]

where \( |C(r_m)| \) is the measure of the set \( C(r_m) \) defined in (14), (17). Now (20), (2.5), and (2.6) imply

(2.7)

\[
1 \leq \sigma(1 + o(1)) \left\{ \frac{(q+1)}{\sin \pi \mu} \sin \left( \frac{\mu}{q+1} |C(r_m)| \right) \right\} + o(1)
\]

as \( m \to \infty \).

We may assume that for all large \( m \), \( |C(r_m)| < (q+1)\pi/2\mu \), since otherwise the inequality in (11) is manifestly true. Thus (2.7) now implies that

(2.8)

\[
|C(r_m)| \geq \frac{q+1}{\mu} \sin^{-1} \left\{ \frac{\sin \pi \mu}{(q+1)} \cdot \frac{1 + o(1)}{\sigma} \right\} \quad (m \to \infty).
\]
Recalling (19) and (6), we see that (2.8) leads to
\[
\liminf_{m \to \infty} |E(r_m)| \geq \frac{2(q + 1)}{\mu} \sin^{-1} \left\{ \frac{|\sin \pi \mu|}{(q + 1)(1 - \delta(0))} \right\},
\]
and the proof of (11) is complete when \( q \) is odd.

If \( q \) is an even integer, we have to proceed slightly differently: Since \( [(q + 1)/2] = q/2 \), the first sum in (2.5) becomes
\[
\frac{q}{2} \csc \pi \mu \sum_{j=0}^{q/2} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j})
\]
\[
= \frac{q}{2} \csc \pi \mu \left\{ \sin \mu \alpha_1 + \sum_{j=1}^{q/2} 2 \cos \frac{1}{2} \mu (\alpha_{2j+1} + \alpha_{2j}) \sin \frac{1}{2} \mu (\alpha_{2j+1} - \alpha_{2j}) \right\}
\]
\[
\leq \frac{q}{2} \csc \pi \mu \left\{ \sin \mu \alpha_1 + 2 \sum_{j=1}^{q/2} \sin \frac{1}{2} \mu (\alpha_{2j+1} - \alpha_{2j}) \right\},
\]
where the inequality is justified as in the case of odd \( q \). We now let \( x_0 = \mu \alpha_1 \), \( x_j = \frac{1}{2} \mu (\alpha_{2j+1} - \alpha_{2j}) \) for \( j = 1, \ldots, q/2 \) and \( x_{-j} = x_j \). This gives us \( q + 1 \) numbers in \([0, \pi]\) to which we may apply the concavity of \( \sin x \). We thus obtain
\[
\sin \left\{ (q + 1)^{-1} \left( x_0 + \sum_{j=1}^{q/2} (x_j + x_{-j}) \right) \right\} \geq (q + 1)^{-1} \left\{ \sin x_0 + 2 \sum_{j=1}^{q/2} \sin x_j \right\}.
\]
It follows that
\[
\sin \mu \alpha_1 + 2 \sum_{j=1}^{q/2} \sin \frac{1}{2} \mu (\alpha_{2j+1} - \alpha_{2j})
\]
\[
\leq (q + 1) \sin \left\{ \frac{\mu \alpha_1 + \sum_{j=1}^{q/2} \mu (\alpha_{2j+1} - \alpha_{2j})}{q + 1} \right\}
\]
\[
= (q + 1) \sin \left\{ \frac{\mu}{q + 1} |C(r_m)| \right\}.
\]
Now (2.10) and (2.11) lead us back to (2.8) and (2.9). Hence (11) is true when \( q \) is odd, and the proof of Theorem 1 is complete except for the statement about the sharpness of the inequalities.

3. Proof of Theorem 2

Assume that the lower order \( \mu \) of \( g \) satisfies
\[
q < \mu \leq q + \frac{1}{2},
\]
and that for infinitely many values of \( m \), \( \pi \in E(r_m) \). By taking a subsequence of \( r_m \) which we also denote by \( r_m \), we may assume that \( \pi \in E(r_m) \) for all \( m \). This means that we have

\[
\alpha_{q+1}(r_m) = \pi \quad \text{for all } m.
\]

Now suppose that \( q \) is even. Since \( \alpha_0 = 0 \), we may write

\[
|\csc \pi \mu| \sum_{j=0}^{[q/2]} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j})
\leq |\csc \pi \mu| \left\{ 2 \sum_{j=1}^{q/2} \sin \frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}) + \sin \pi \mu \right\}
\leq 1 + 2|\csc \pi \mu|(q/2) \sin \left\{ \frac{1}{q} \sum_{j=1}^{q/2} \frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}) \right\}
= 1 + q|\csc \pi \mu| \sin \left\{ \frac{\mu}{q} \sum_{j=1}^{q/2} (\alpha_{2j} - \alpha_{2j-1}) \right\}.
\]

Recalling (18) and (3.2) we have, with \( r = r_m \),

\[
\sum_{j=1}^{q/2} (\alpha_{2j} - \alpha_{2j-1}) = \alpha_{q+1} - |C(r)| = \pi - |C(r)|.
\]

Now (3.3) and (3.4) used in (20) give

\[
1 \leq \sigma(1 + o(1)) \cdot \left( 1 + q|\csc \pi \mu| \sin \left\{ \frac{\pi \mu}{q} - \frac{\mu}{q} |C(r_m)| \right\} \right) + o(1)
\]

as \( m \to \infty \).

Since \( 0 \leq y \leq \sin x \) for some \( x \) in \([0, \pi]\) implies \( \sin^{-1} y \leq \pi - x \), we conclude from (3.5), with \( x = \frac{\mu}{q} (\pi - |C(r_m)|) \), that

\[
\sin^{-1} \left\{ \frac{|\sin \pi \mu|}{q} \left( \frac{1 + o(1)}{\sigma(1 + o(1))} - 1 \right) \right\} \leq \pi - (\pi - |C(r_m)|) \frac{\mu}{q};
\]

and this readily yields (13) when \( q \) is even.

If \( q \) is odd, then (17) and (3.2) imply that

\[
\sum_{j=1}^{(q+1)/2} (\alpha_{2j} - \alpha_{2j-1}) = \pi - \sum_{j=0}^{(q-1)/2} (\alpha_{2j+1} - \alpha_{2j}).
\]
Since $\alpha_0 = \alpha_{q+2} = 0$, the sum appearing in (20) may be written

$$\left| \csc \pi \mu \right| \sum_{j=0}^{[(q+1)/2]} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j})$$

$$= \left| \csc \pi \mu \right| \left\{ \sin \mu \alpha_1 + \sum_{j=1}^{(q-1)/2} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j}) + |\sin \pi \mu| \right\}$$

$$\leq \left| \csc \pi \mu \right| \left\{ \sin \mu \alpha_1 + 2 \sum_{j=1}^{(q-1)/2} \sin \frac{1}{2} \mu (\alpha_{2j+1} - \alpha_{2j}) + |\sin \pi \mu| \right\}$$

$$\leq \left| \csc \pi \mu \right| \left\{ q \sin \left( \frac{\mu \alpha_1 + 2 \sum_{j=1}^{(q-1)/2} \frac{1}{2} \mu (\alpha_{2j+1} - \alpha_{2j})}{q} \right) + |\sin \pi \mu| \right\}$$

$$= \left| \csc \pi \mu \right| \left\{ q \sin \left( \frac{\mu \alpha_1 + 2 \sum_{j=1}^{(q-1)/2} \frac{1}{2} \mu (\alpha_{2j+1} - \alpha_{2j})}{q} \right) + \frac{\mu \pi}{q} |C(r_m)| \right\}$$

Thus the inequality (3.5) also holds true in the case of odd $q$, and hence (13) follows in this case also. Note that if $q = 1$, the sum in the second line in (3.7) is empty.

4. THE BEST POSSIBLE CHARACTER OF THEOREMS 1 AND 2

Let $g$ be a Weierstrass canonical product of genus $q \geq 1$ and order $\mu$, $q < \mu < q + 1$, having only real negative zeros and satisfying

$$N(r) = N(r, 1/g) \sim r^\mu L(r) \quad \text{as} \ r \to \infty,$$

where $L$ is a slowly varying function. It is known [1] that

$$\lim_{r \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\log |g(re^{i\theta})|}{N(r)} - \psi_\mu(\theta) \right|^p \ d\theta = 0 \quad (1 \leq p < \infty).$$

It follows from (4.2) that

$$\lim_{r \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\log^+ |g(re^{i\theta})|}{N(r)} - \psi^+_\mu(\theta) \right| \ d\theta = 0,$$

and also that

$$\lim_{r \to \infty} \left| \int_{E(r)} \left( \frac{\log |g(re^{i\theta})|}{N(r)} - \psi_\mu(\theta) \right) \ d\theta \right| = 0,$$

where $E(r)$ is any measurable subset of $[-\pi, \pi]$. 
Now (4.3) implies that \( \lim_{r \to \infty} T(r, g)/N(r) = \frac{1}{\pi} \int_0^{\pi} \psi_\mu^+(\theta) \, d\theta \). If we take \( E(r) \) to be the set defined in (4), then (4.4) implies

\[
\lim_{r \to \infty} \phi(E(r)) = \lim_{r \to \infty} \frac{1}{\pi} \int_{E(r)} \log \left| g(re^{i\theta}) \right| \frac{d\theta}{N(r)} = \frac{2}{1 - \delta(0)},
\]

and this shows that equality can hold in (10).

From (4.5) and (4.1), it follows that \( T(r, g) \) is regularly varying and so every sequence is a sequence of Polya peaks of order \( \mu \) of \( T(r, g) \). Thus, by the remark after the proof of (2.4), we conclude that for such functions \( g \),

\[
\alpha_j(r) \to \frac{(2j - 1)\pi}{2\mu} \quad \text{as} \quad r \to \infty
\]

for \( j = 1, \ldots, (q + 1) \) if \( q + \frac{1}{2} \leq \mu < q + 1 \); and for \( j = 1, \ldots, q \) and \( \alpha_{q+1}(r) \to \pi \) if \( q < \mu \leq q + \frac{1}{2} \). It follows that for such functions \( g \), the measure of the set \( E(r) \) defined in (4) satisfies

\[
|E(r)| \to |E_\mu|,
\]

where \( E_\mu = \{ \theta \in [-\pi, \pi] : \psi_\mu(\theta) \geq 0 \} \).

We can also compute \( \delta(0) \):

\[
\frac{1}{1 - \delta(0)} = \frac{1}{\pi} \int_0^{\pi} \psi_\mu^+(\theta) \, d\theta = \begin{cases} 
\frac{q + 1}{|\sin \pi \mu|} & \text{if } q + \frac{1}{2} \leq \mu < q + 1 \\
\frac{q + |\sin \pi \mu|}{|\sin \pi \mu|} & \text{if } q < \mu \leq q + \frac{1}{2}.
\end{cases}
\]

It follows that

\[
|E_\mu| = \frac{(q + 1)\pi}{\mu} = \frac{2(q + 1)}{\mu} \sin^{-1} \left\{ \frac{|\sin \pi \mu|}{(q + 1)(1 - \delta(0))} \right\} \quad \text{if } q + \frac{1}{2} < \mu < q + 1.
\]

And if \( q < \mu < q + \frac{1}{2} \), then

\[
|E_\mu| = \frac{(q - 1)\pi}{\mu} + 2 \left( \pi - \frac{(2q - 1)\pi}{2\mu} \right) = 2\pi \left( 1 - \frac{q}{\mu} \right) + \frac{(q - 1)\pi}{\mu} + \frac{\pi}{\mu} = 2\pi \left( 1 - \frac{q}{\mu} \right) + \frac{q}{2\mu} \sin^{-1} \left( \frac{|\sin \pi \mu|\delta(0)}{q(1 - \delta(0))} \right).
\]

Now (4.7) and (4.8) imply that equality can hold in (13), which must therefore be a best possible inequality.

**References**


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