ON CONTRACTIONS WITHOUT DISJOINT INVARIANT SUBSPACES

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Abstract. Assume $T$ is a contraction with the following property: there exists an operator $X$ with dense range such that $XT = WX$ where $W$ is a bilateral shift. We give a necessary and sufficient condition that $T$ has no disjoint invariant subspaces.

In [3] and [4], Olin and Thomson investigated subnormal operators without disjoint invariant subspaces. They called a (bounded linear) operator $T$ on a Hilbert space cellular-indecomposable if the intersection of any two nonzero invariant subspaces of $T$ is nonzero. In this note we extend the result for cellular-indecomposable subnormal contractions proved in [4].

Let $T$ be a contraction on a separable Hilbert space $H$, and suppose that $T$ is not of class $C_0$; that is, $\lim_{n \to \infty} \|T^n x\| \neq 0$ for some $x \in H$. It follows that there exist an isometry $V$ and an operator $X$ with dense range such that $XT = VX$ (see [5, Proposition II.3.5]). Kerchy [2] proved that if this $V$ is a bilateral shift, then $T$ has a nontrivial invariant subspace. (Note that if $V$ is not unitary, then it can be replaced by a bilateral shift.)

Theorem. Let $T$ be a contraction on $H$ and assume that there exists an operator $X$ with dense range such that $XT = WX$ where $W$ is the bilateral shift on $L^2$. Then $T$ is cellular-indecomposable if and only if there exists a quasiaffinity (i.e. an injection with dense range) $Y$ such that $YT = SY$ where $S$ is the unilateral shift on $H^2$.

The existence of the operator $X$ in the theorem is equivalent to the condition that $\Theta_T(\zeta)^*$ is not isometric a.e. on the unit circle $\mathbb{T}$ where $\Theta_T$ is the characteristic function of $T$ (see [7]). If $T$ is a contraction such that $YT = SY$ where $Y$ is a quasiaffinity and $S$ is the unilateral shift, then the restriction of $T$ to any of its cyclic invariant subspaces is quasisimilar to $S$ (see [1, Corollary 15]). Therefore our theorem extends the result of [4]. Such contractions were also considered in [8].

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Proof of theorem. Assume that there exists a quasiaffinity $Y$ such that $YT = SY$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two nonzero invariant subspaces of $T$. For $i = 1, 2$, there exists an injection $K_i$ such that $K_iS = TK_i$ and $\text{ran } K_i \subseteq \mathcal{M}_i$ (see [1, Lemma 3]). Since $YK_i$ commutes with $S$, we have $YK_i = f_i(S)$ for a nonzero function $f_i \in H^\infty$. (For a contraction $A$ whose unitary part is absolutely continuous or acts on the space $\{0\}$ and $f \in H^\infty$, $f(A)$ is an operator obtained by the Sz.-Nagy and Foias functional calculus [5]). It follows from the injectivity of $Y$ that $Y(\mathcal{M}_1 \cap \mathcal{M}_2) = Y\mathcal{M}_1 \cap Y\mathcal{M}_2$. Therefore we have

$$Y(\mathcal{M}_1 \cap \mathcal{M}_2) \supseteq YK_1H^2 \cap YK_2H^2 = f_1H^2 \cap f_2H^2 \supseteq f_1f_2H^2 \neq \{0\},$$

and $\mathcal{M}_1 \cap \mathcal{M}_2 \neq \{0\}$.

Conversely, we assume that $T$ is cellular-indecomposable. Obviously $T$ is completely nonunitary. By the proof of the theorem in [7] (see [2] for a special case), it follows from the existence of the operator $X$ that there are an operator $Z$ and a vector $x_0$ satisfying the following conditions:

1. $ZT = WZ$ and
2. the function $\log |Zx_0|$ is integrable.

The condition (2) implies $Zx_0 = ug$ where $|u| = 1$ a.e. and $g$ is an outer function in $H^2$, and so $W(Z\mathcal{M}_0)^-$ is a unilateral shift of multiplicity one where $\mathcal{M}_0 = \bigvee_{n \geq 0} T^n x_0$. Since $Z|\mathcal{M}_0$ is injective (see the proof of [1, Corollary 15]), $\ker Z \cap \mathcal{M}_0 = \{0\}$. Since the subspaces $\ker Z$ and $\mathcal{M}_0$ are invariant under $T$ and $T$ is cellular-indecomposable, it follows that $\ker Z = \{0\}$.

We claim that for every $x \in \mathcal{H}$, there exists a nonzero $f \in H^\infty$ such that $Zf(T)x \in (Z\mathcal{M}_0)^-$. For this purpose, take $x \in \mathcal{H}$ and let $\alpha = \{\zeta \in T: |(Zx)(\zeta)| \geq |(Zx_0)(\zeta)|\}$. Let us choose a function $h \in H^\infty$ such that $|h(\zeta)| = 1/2$ for $\zeta \in \alpha$ and $|h(\zeta)| = 2$ for $\zeta \in T\setminus\alpha$. Then on $\alpha$ we have

$$|Zx + hZx_0| \geq |Zx| - |h||Zx_0| \geq (1 - |h|)|Zx_0| = (1/2)|Zx_0|,$$

and on $T\setminus\alpha$ we have

$$|Zx + hZx_0| \geq |h||Zx_0| - |Zx| \geq (|h| - 1)|Zx_0| = |Zx_0|.$$

Thus $\log |Zx + hZx_0|$ is integrable, and so $Zx + hZx_0 = vk$ where $|v| = 1$ a.e. and $k$ is outer. Then, setting $\mathcal{M} = \bigvee_{n \geq 0} T^n(x + h(T)x_0)$, we have

$$uH^2 \cap vH^2 \supseteq Z\mathcal{M}_0 \cap Z\mathcal{M} \supseteq Z(\mathcal{M}_0 \cap \mathcal{M}) \neq \{0\}$$

because $T$ is cellular-indecomposable. Thus there exists a nonzero function $f \in H^\infty$ such that $ufv \in H^\infty$. Then we have

$$Zf(T)x = fZx = f(vk - hZx_0) = u(ufvk - hf) \in uH^2 = (Z\mathcal{M}_0)^-.$$

This establishes the claim.
Let \( \mathcal{N} = \{ x \in \mathcal{H} : Zx \in (Z\mathcal{M}_0)^- \} \), which is an invariant subspace of \( T \). Let \( T_1 = PT| \mathcal{H} \ominus \mathcal{N} \) where \( P \) is the orthogonal projection onto \( \mathcal{H} \ominus \mathcal{N} \). The claim given above implies that for every \( x \in \mathcal{H} \ominus \mathcal{N} \), there exists a nonzero function \( f \in H^\infty \) such that \( f(T_1)x = 0 \), and so \( T_1 \) is of class \( C_0 \) by [6]. That is, there exists an inner function \( q \) such that \( q(T_1) = 0 \). Then \( q(W)Z\mathcal{H} \subseteq (Z\mathcal{M}_0)^- \). Therefore \( W|(q(W)Z\mathcal{H})^- \) is unitarily equivalent to the unilateral shift \( S \). Since \( q(W)Z \) is injective, it follows that there is a quasiaffinity \( Y \) such that \( YT = SY \).

REFERENCES

6. ——, Local characterization of operators of class \( C_0 \), J. Funct. Anal. 8 (1971), 76–81.

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