PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 110, Number 4, December 1990

THE EXTENSION OF THE THEOREMS OF Č. V. STANOJEVIČ AND V. B. STANOJEVIČ

SHENG SHUYUN

(Communicated by J. Marshall Ash)

ABSTRACT. The new necessary-sufficient conditions for L^1 convergence of Fourier series are obtained; Č. V. Stanojevič and V. B. Stanojevič's theorem [5] and Singh and Sharma's theorem [2] are modified; the convergence theorem for a function sequence in L^1 space is obtained; and the extensions are made for the Sheng and Yang theorem [6] and Singh and Sharma's results.

1. INTRODUCTION

Let $\{q_n\}$ be a monotone decreasing sequence for sufficiently large n. If the limit of q_n is zero as $n \to \infty$, then it is denoted by $q_n \downarrow 0$. If there exists $\beta > 0$ such that $n^{-\beta} q_n \downarrow 0$, then the sequence $\{q_n\}$ is called a quasi-monotone sequence and is denoted by $q_n \,\mathring{\downarrow}\, 0$. It is obvious that $q_n \downarrow 0$ implies $q_n \,\mathring{\downarrow}\, 0$.

Assume $\{C_n\} = \{a_n + ib_n\}$ is a null sequence of complex numbers. A complex null sequence $\{C_n\}$ satisfying $\sum_{n=1}^{\infty} |\Delta(C_n - C_{-n})| \lg n < \infty$, is called weakly even and is denoted by $C_n \in W$. It is obvious that if $\{C_n\}$ is an even sequence then it is weakly even. If $\{C_n\}$ is satisfying $\sum_{n=1}^{\infty} |\Delta(C_n - C_{-n})| n^r \lg n < \infty$, $r = 0, 1, 2, \dots$, then it is denoted by $C_n \in W_r$, where $W_0 = W$.

For convenience, the following notations are used:

$$M = \left\{ A_n | A_n \downarrow 0 \text{ and } \sum_{n=1}^{\infty} A_n < \infty \right\},$$

$$M_\alpha = \left\{ A_n | A_n \downarrow 0 \text{ and } \sum_{n=1}^{\infty} n^\alpha A_n < \infty \text{ for some } \alpha \geq 0 \right\},$$

$$S = \left\{ a_n | a_n \to 0,\ n \to \infty,\ |\Delta a_n| \leq A_n \forall n, \text{ and } A_n \in M \right\},$$

$$S_{p\alpha} = \left\{ a_n | a_n \to 0,\ n \to \infty,\ (1/n) \sum_{k=1}^{n} |\Delta a_k| A_k^p = O(1),\ 1 < p \leq 2, \text{ and, for some } \alpha \geq 0,\ A_n \in M_\alpha \right\},$$

Received by the editors December 12, 1988 and, in revised form, September 20, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42A16; Secondary 42A20.

$$S_p^* = \left\{ C_n | C_n \in W, \ (1/n)\sum_{k=1}^{n} |\Delta a_k|/A_k^p = O(1), \ 1 < p \le 2, \text{ and } A_n \in M \right\},$$

and

$$S_{p\alpha r}^* = \left\{ C_n | \text{ for some } \alpha \ge 0, \ r \in \{0, 1, \dots, [\alpha]\}, \right.$$

$$\left. C_n \in W_r, \ 1/n^{p(\alpha - r)+1} \sum_{k=1}^{n} |\Delta C_k|^p / A_k^p = O(1), \ 1 < p \le 2, \text{ and } A_n \in M_\alpha \right\}.$$

When $\alpha = 0$, denote $M_\alpha = M_0$, $S_\alpha = S_0$, and when $r = \alpha = 0$, denote $S_{p\alpha r}^* = S_{poo}^*$. It is obvious that $M_\alpha \supset M_0 \supset M$, $S_{p\alpha r}^* \supset S_{poo}^* \supset S_p^*$, and $S_{p\alpha r}^* \supset S_{p\alpha} \supset S_\alpha \supset S_0 \supset S$.

Telyakovskii [7] proved that if $a_n \in S$ then the series $a_0/2 + \sum_{k=1}^{\infty} a_k \cos kx$ is the Fourier series of its sum f, and $\|S_n(f) - f\| = o(1)$, $n \to \infty$, is equivalent to $a_n \lg n = o(1)$, where $S_n(f) = S_n(f, x) = a_0/2 + \sum_{k=1}^{\infty} a_k \cos kx$ and $\|\cdot\|$ denotes the $L^1(0, \pi)$-norm (a notation to be used throughout the paper). Singh and Sharm [2] showed that the condition $a_n \in S$ can be reduced to be $a_n \in S_0$; Sheng and Yang [6] showed that the condition again can be reduced to be $a_n \in S_\alpha$; in this paper, we are going to show that the condition can be further reduced to be $a_n \in S_{p\alpha}$.

The partial sums of the complex trigonometric series $\sum_{|n|<\infty} C_n e^{int}$ will be denoted by $S_n(c) = S_n(c, t) = \sum_{|k| \le n} C_k e^{ikt}$, $t \in T = R/2\pi Z$. If a trigonometric series is the Fourier series of some $f(t) \in L^1$, we will write $C_n = \hat{f}(n)$, for all n, and $S_n(c, t) = S_n(f, t) = S_n(f)$. Č. V. Stanojeviĉ and V. B. Stanojeviĉ [5] studied this problem from another aspect and improved Telyakovskii's results. They proved that if $C_n \in S_p^*$ then, for $t \ne 0$, $\lim_{n\to\infty} S_n(C, t) = f(t)$ exists and $f \in L^1(T)$ and that $\|S_n(f) - f\| = o(1)$, $n \to \infty$, is equivalent to $\hat{f}(n) \lg |n| = o(1)$, $|n| \to \infty$. In this paper we will prove that the condition $C_n \in S_p^*$ can be reduced to be $C_n \in S_{p\alpha r}^*$; also, we will improve the Singh and Sharma's result [1] about L^1 convergence.

2. Lemmas

Lemma 2.1. *Let* $A_n \in M_\alpha$, $\alpha \ge 0$. *Then* $\sum_{n=1}^{\infty} n^{1+\alpha} |\Delta A_n| < \infty$.

Proof. It follows from [3] and [4] that $A_n = 0 \ (n^{-1-\alpha})$ and $|\Delta A_n| = \Delta A_n + O(A_n/n)$. Then

$$\begin{aligned}\sum_{k=1}^{n-1} k^{1+\alpha} |\Delta A_k| &= \sum_{k=1}^{n-1} k^{1+\alpha} \Delta A_k + \sum_{k=1}^{n-1} O(k^\alpha A_k) \\ &= \sum_{k=1}^{n} [k^{1+\alpha} - (k-1)^{1+\alpha}] A_k - n^{1+\alpha} A_n + \sum_{k=1}^{n-1} O(k^\alpha A_k) \\ &= \sum_{k=1}^{n} O(k^\alpha A_k) + O(1).\end{aligned}$$

This completes the proof. □

Lemma 2.2. *Let r be a nonnegative integer, and $x \in [\pi/n, \pi]$ where $n \geq 1$. Then*

$$D_n^{(r)}(\alpha) = \sum_{k=0}^{r-1} \frac{(n+1/2)^k \sin[(n+1/2)x + k\pi/2]}{(\sin(x/2))^{r+1-k}} \varphi + \frac{(n+1/2)^r \sin[(n+1/2)x + r\pi/2]}{2\sin x/2}, \tag{2.1}$$

where the same φ denotes various analytical functions of x independent of n, and $D_n(x)$ is the Dirichlet kernel.

Proof. The proof is straightforward in the case of $r = 0$. Assuming that Equation (2.1) holds, and taking the derivatives of both sides in this equation, we have

$$\begin{aligned} D_n^{(r+1)}(x) = \sum_{k=0}^{r-1} \Bigg\{ & (n+1/2)^{k+1} \sin[(n+1/2)x + (k+1)\pi/2](\sin(x/2))^{k-r-1} \varphi \\ & + (n+1/2)^k \sin[(n+1/2)x + k\pi/2](\sin(x/2))^{k-r-2} \varphi \\ & + \frac{(n+1/2)^{r+1} \sin[(n+1/2)x + (r+1)\pi/2]}{2\sin(x/2)} \\ & + \frac{(n+1/2)^r \sin[(n+1/2)x + r\pi/2]}{\sin^2(x/2)} \varphi \Bigg\} \\ = \sum_{r=0}^{r} & (n+1/2)^k \sin[(n+1/2)x + k\pi/2] \cdot (\sin(x/2))^{k-r-2} \varphi \\ & + (n+1/2)^{r+1} \sin[(n+1/2)x + (r+1)\pi/2] 2(\sin(x/2))^{-1}. \end{aligned}$$

The proof follows by induction. □

Lemma 2.3. *Let $n \geq 1$, and let r be a nonnegative integer, $x \in [\varepsilon, \pi]$. Then*

$$|D_n^{(r)}(x)| \leq C_\varepsilon n^r / x,$$

where C_ε is a positive constant depending on ε, and $0 < \varepsilon < \pi$.

Proof. This is a direct result from Lemma 2.2. □

Lemma 2.4. $\|D_n^{(r)}(x)\| = (4/\pi) n^r \lg n + O(n^r)$, $r \in \{0, 1, \dots\}$.

Proof. It follows from $D_n^{(r)}(x) = O(n^{r+1})$ and Lemma 2.2 that

$$\begin{aligned} \|D_n^{(r)}(x)\| &= 2\left\{ \int_0^{\pi/n} + \int_{\pi/n}^{\pi} \right\} |D_n^{(r)}(x)|\, dx \\ &= 2n^r \int_{\pi/n}^{\pi} \frac{|\sin(nx + r\pi/2)|}{x}\, dx + \sum_{k=0}^{n-1} \int_{\pi/n}^{\pi} n^k x^{k-1-r}\, dx + O(n^r) \\ &= \frac{4}{\pi} n^r \lg n + O(n^r). \quad \square \end{aligned}$$

Lemma 2.5. $\|\overline{D}_n^{(r)}(x)\| = O(n^r \lg n)$, $r \in \{0, 1, \dots\}$.

Proof. With the Bernstein inequality in L space, we have

$$\int_0^\pi |\overline{D}_n^{(r)}(x)|\,dx \le n^r \int_0^\pi |\overline{D}_n(x)|\,dx. \tag{2.2}$$

On the other hand,

$$\int_0^\pi |\overline{D}_n(x)|\,dx \le \int_0^\pi \frac{\sin^2(nx/2)}{x}\,dx + O(1)$$
$$= \lg(1 + n\pi) + O(1) = O(\lg n).$$

By combining Equations (2.2) and (2.3) the proof is completed. □

Lemma 2.6. *For each nonnegative integer n, there holds*

$$\|C_n E_n^{(r)}(t) + C_{-n} E_{-n}^{(r)}(t)\| = o(1) \Leftrightarrow n^r C_n \lg|n| = o(1),$$

where $\{C_n\}$ is a complex sequence, $E_n(t) = \sum_{k=0}^n e^{ikt}$ and $t \in R/2\pi Z$.

Proof. Assuming $r \ge 1$ and denoting $J_n = \|C_n E_n^{(r)}(t) + C_{-n} E_{-n}^{(r)}(t)\|$, from Lemma 2.4 we have

$$\begin{aligned} J_n &= \int_0^\pi \{|C_n E_n^{(r)}(t) + C_{-n} E_{-n}^{(r)}(t)| + |C_n E_{-n}^{(r)}(t) + C_{-n} E_n^{(r)}(t)|\}\,dt \\ &\ge |C_n + C_{-n}| \int_0^\pi |E_n^{(r)}(t) + E_{-n}^{(r)}(t)|\,dt \\ &= |C_n + C_{-n}| \int_0^\pi 2|D_n^{(r)}(t)|\,dt \ge \frac{4}{\pi}|C_n + C_{-n}| n^r \lg n + O(1). \end{aligned} \tag{2.4}$$

On the other hand, using

$$\begin{aligned} J_n &= \int_{-\pi}^\pi |(C_n + C_{-n}) E_n^{(r)}(t) + C_{-n}(E_{-n}^{(r)}(t) - E_n^{(r)}(t)|\,dt \\ &\le |C_n + C_{-n}| \int_{-\pi}^\pi |E_n^{(r)}(t)|\,dt + |C_{-n}| \int_{-\pi}^\pi |E_{-n}^{(r)}(t) - E_n^{(r)}(t)|\,dt \end{aligned} \tag{2.5}$$

with Lemma 2.4 and Lemma 2.5, the right-hand side of Equation (2.5) can be written as

$$O\{|C_n + C_{-n}| n^r \lg n\} + O\{|C_{-n}| n^r \lg n\} = O\{|C_n + C_{-n}| n^r \lg n\}. \tag{2.6}$$

Combining Equations (2.4), (2.5), and (2.6) completes the proof. □

3. L^1 CONVERGENCE OF FOURIER SERIES AND FUNCTION SEQUENCES IN L^1 SPACE

Let $S_n(C, t) = \sum_{|k|\le n} C_k e^{ikt}$, $t \in T = R/2\pi Z$. The limit of $S_n(C, t)$ is denoted by $f(t)$. If $f(t) \in L$, then $\hat{f}(k) = C_k$ is called the Fourier coefficient of $f(t)$. Let

$$\lim_{n\to\infty} S_n^{(r)}(C, t) = f_r(t), \qquad r \in \{1, 2, \dots\}.$$

If $f_r(t) \in L$, then it is denoted by $f^{(r)}(t)$.

Theorem 3.1. (main theorem). *Let* $C_n \in S^*_{p\alpha r}$, $\alpha \geq 0$, *and* $r \in \{0, 1, \ldots [\alpha]\}$. *Then, for* $t \neq 0$,

(i) $\lim_{n\to\infty} S_n^{(r)}(C, t) = f_r(t)$,
(ii) $f_r(t) \in L(T)$,
(iii) $\|S_n^{(r)}(f, t) - f^{(r)}(t)\| = o(1) \Leftrightarrow n^r \hat{f}(n) \lg|n| = o(1)$.

Proof. It can be shown that

$$\sum_{k=1}^{n} |\Delta[(ik)^r C_k]| = \sum_{k=1}^{n} |(ik)^r \Delta C_k + C_{k+1}\Delta[(ik)^r]| \leq \sum_{k=1}^{n} k^r |\Delta C_k| + \sum_{k=1}^{n} k^{r-1}|C_{k+1}|.$$

Notice that

$$\sum_{k=1}^{n} k^{r-1}|C_{k+1}| = O\left\{\sum_{k=1}^{n-1} |\Delta C_{k+1}| k^r + \sum_{k=n+1}^{\infty} k^r |\Delta C_k|\right\}$$

and

$$\sum_{k=1}^{n} k^r|\Delta C_k| = \sum_{k=1}^{n-1} \Delta A_k \sum_{j=1}^{k} \frac{\Delta C_j}{A_j} j^r + A_n \sum_{j=1}^{n} \frac{|\Delta C_j|}{A_j} j^r$$
$$\leq \sum_{k=1}^{n-1} \Delta A_k k^{1+\alpha} \left(k^{p(r-\alpha)-1} \sum_{j=1}^{k} \frac{|\Delta C_j|^p}{A_j^p}\right)^{1/p} + n^{1+\alpha} A_n \left(n^{p(r-\alpha)-1} \sum_{j=1}^{n} \frac{|\Delta C_j|^p}{A_j^p}\right)^{1/p}$$
$$= O(1)$$

Therefore $\sum_{k=1}^{\infty} |\Delta(ik)^r C_k| < \infty$, and (i) is proved.

Now the proof of (ii): it is obvious for the case of $r = 0$. Assuming $r \geq 1$, it can be shown that

$$S_n^{(r)}(C, t) - (C_n E_n^{(r)}(t) + C_{-n} E_{-n}^{(r)}(t))$$
$$= \sum_{k=0}^{n} C_k (ik)^r e^{ikt} + \sum_{k=1}^{n} (C_{-k} - C_k)(-ik)^r e^{-ikt}$$
$$+ \sum_{k=1}^{n} C_k (-ik)^r e^{-ikt} - C_n E_n^{(r)}(t) - C_{-n} E_{-n}^{(r)}(t)$$
$$= 2\left(\sum_{k=0}^{n-1} \Delta C_k D_k^{(r)}(t)\right) + \sum_{k=1}^{n-1} \Delta(C_{-k} - C_k) E_{-k}^{(r)}(t) \triangleq g_{n,t}(t).$$

For $t \neq 0$, it follows from (i) that

$$\begin{aligned} f_r(t) - g_{n,r}(t) &= \sum_{k=0}^{\infty} C_k (e^{ikt})^{(r)} + \sum_{k=1}^{\infty} C_{-k}(e^{-ikt})^{(r)} - g_{n,t}(t) \\ &= 2\sum_{k=0}^{\infty} \Delta C_k D_k^{(r)}(t) + \sum_{k=1}^{\infty} \Delta(C_{-k} - C_k)(E_{-k}^{(r)}(t)) \\ &\quad - 2\sum_{k=0}^{n-1} \Delta C_k D_k^{(r)}(t) - \sum_{k=0}^{n-1} \Delta(C_{-k} - C_k)(E_{-k}^{(r)}(t)) \\ &= 2\sum_{k=r}^{\infty} \Delta C_k D_k^{(r)}(t) + \sum_{k=n}^{\infty} \Delta(C_{-k} - C_k)(E_{-k}^{(r)}(t)). \end{aligned}$$

From the above results and Lemmas 2.4, 2.5 and $C_n \in S^*_{p\alpha r}$, we can get

$$\begin{aligned} \|f_r(t) - g_{n,r}(t)\| &\le 2\int_0^{\pi} \left|\sum_{k=n}^{\infty} \Delta C_k D_k^{(r)}(t)\right| dt \\ &\quad + O\left\{\sum_{k=n}^{\infty} \left|\Delta(C_{-k} - C_k)\right| \int_0^{\pi} |E_{-t}^{(r)}(t)|\, dt\right\} \\ &\quad + O(1) = O\left\{\int_0^{\pi} \left|\sum_{k=n}^{\infty} \Delta C_k D_k^{(r)}(t)\right| dt + O(1)\right\} \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &\int_0^{\pi} \left|\sum_{k=n}^{\infty} \Delta C_k D_k^{(r)}(t)\right| dt \\ &\quad \le \sum_{k=n}^{\infty} |\Delta A_k| \int_0^{\pi} \left|\sum_{j=1}^{k} \frac{\Delta C_j}{A_j} D_j^{(r)}(t)\right| dt + A_n \int_0^{\pi} \left|\sum_{j=1}^{n-1} \frac{\Delta C_j}{A_j} D_j^{(r)}(t)\right| dt \\ &\quad = I_1 + I_2, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} I_1 &= \sum_{k=n}^{\infty} |\Delta A_k| \int_0^{\pi/k} \left|\sum_{j=1}^{k} \frac{\Delta C_j}{A_j} D_j^{(r)}(t)\right| dt + \sum_{k=n}^{\infty} |\Delta A_k| \int_{\pi/k}^{\pi} \left|\sum_{j=1}^{k} \frac{\Delta C_j}{A_j} D_j^{(r)}(t)\right| dt \\ &= I_{11} + I_{12}. \end{aligned} \tag{2.9}$$

It follows from the hypotheses of Theorem 3.1 and Lemma 2.1 that

$$\begin{aligned} I_{11} &= O\left\{\sum_{k=n}^{\infty} |\Delta A_k| k^{1+\alpha} \left(k^{p(r-\alpha)-1} \sum_{j=1}^{k} \frac{|\Delta C_j|^p}{A_j^p}\right)^{1/p}\right\} \\ &= O\left\{\sum_{k=n}^{\infty} k^{1+\alpha} |\Delta A_k|\right\} = O(1). \end{aligned} \tag{2.10}$$

Now we estimate I_{12}:

$$(2.11)\quad I_{12} = O\left\{\sum_{k=n}^{\infty} |\Delta A_k| \int_{\pi/k}^{\pi} \left|\sum_{j=1}^{k} \frac{\Delta C_j}{A_j} D_j^{(r)}(t)\right| dt\right\} \triangleq O\left\{\sum_{k=n}^{\infty} |\Delta A_k| \varphi_k(t)|\right\}.$$

From Lemma 2.2, we have

(2.12)

$$\begin{aligned}\varphi_k(t) &= \int_{\pi/k}^{\pi} \left|\sum_{j=1}^{k} \frac{|\Delta C_j|}{A_j} D_j^{(r)}(t)\right| dt \\ &= \int_{\pi/k}^{\pi} \left|\sum_{j=1}^{k} \frac{\Delta C_j}{A_j} \left(\sum_{\nu=0}^{r-1} \frac{(j+1/2)^r \sin[(j+1/2)t + \nu\pi/2]}{(\sin(t/2))^{r+1-\nu}} \varphi\right)\right| dt \\ &+ \int_{\pi/k}^{\pi} \left|\sum_{j=1}^{k} \frac{\Delta C_j}{A_j} \frac{(j+1/2)^r \sin[(j+1/2)t + r\pi/2]}{2\sin(t/2)}\right| dt \triangleq \varphi_k'(t) + \varphi_k''(t),\end{aligned}$$

where

$$\varphi_k'(t) = \varphi_{k,1}'(t) + \cdots + \varphi_{k,\nu}'(t) + \cdots + \varphi_{k,r}'(t)$$

$$\varphi_{k,\nu}'(t) = \int_{\pi/k}^{\pi} \left|\sum_{j=1}^{k} \frac{\Delta C_j}{A_j} \frac{(j+1/2)^\nu \sin[(j+1/2)t + \nu\pi/2]}{(\sin(t/2))^{r+1-\nu}} \varphi\right| dt.$$

Since φ is bounded, it can be shown by Hölder's inequality and Hausdorff-Young's inequality that

$$\begin{aligned}\varphi_{k,\nu}'(t) &\le C \int_{\pi/k}^{\pi} \left|\sum_{j=1}^{k} \frac{\Delta C_j}{A_j} \frac{(j+1/2)^\nu \sin[(j+1/2)t + \nu\pi/2]}{(\sin(t/2))^{r+1-\nu}}\right| dt \\ (2.13)\qquad &= O\left\{k^{\frac{(r+1-\nu)p-1}{p}} \left(\sum_{j=1}^{k} \frac{|\Delta C_j|^p}{A_j^p} j^{\nu p}\right)^{1/p}\right\} \\ &= O\left\{\left[k^{p(r-\alpha)-1} \left(\sum_{j=1}^{k} \frac{|\Delta C_j|^p}{A_j^p}\right)\right]^{1/p}\right\} [k^{1+\alpha}].\end{aligned}$$

Since r is a finite value, we have

$$(2.14)\qquad \varphi_k'(t) = O_r\left\{k^{1+\alpha} \left[k^{p(r-\alpha)-1} \left(\sum_{j=1}^{k} \frac{|\Delta C_j|^p}{A_j^p}\right)\right]^{1/p}\right\},$$

where O_r dependents only on r.

Similarly, we can get

$$(2.15)\qquad \varphi_k''(t) = O_r\left\{k^{1+\alpha} \left[k^{p(r-\alpha)-1} \left(\sum_{j=1}^{k} \frac{|\Delta C_j|^p}{A_j^p}\right)\right]^{1/p}\right\}.$$

Combining Equations (2.11)–(2.15), it follows from Lemma 2.1 and the hypotheses of Theorem 3.1 that $I_{12} = o(1)$. From Equations (2.9)–(2.11) we have $I_1 = o(1)$. Similarly, $I_2 = o(1)$ holds. From Equations (2.7) and (2.8), we get $\|f_r(t) - g_{n,r}(t)\| = o(1)$, and $f_r(t) \in L$, since $g_{n,r}(t)$ is a polynomial. Therefore, (ii) is proved.

At last, the proof of (iii). Because

$$f^{(r)}(t) - S_n^{(r)}(f, t) - [\hat{f}(n)E_n^{(r)}(t) + \hat{f}(-n)E_{-n}^{(r)}(t)] = f^{(r)}(t) - g_{n,r}(t),$$

we have

$$\|f^{(r)}(t) - g_{n,r}(t)\| \geq |\|f^{(r)}(t) - S_n^{(r)}(f, t)\| - \|\hat{f}(n)E_n^{(r)}(t) + \hat{f}(-n)E_{-n}^{(r)}(t)\||.$$

It follows from Lemma 2.6 and condition $\|f^{(r)}(t) - g_{n,r}(t)\| = o(1)$ that

$$\|f^{(r)}(t) - S_n^{(r)}(f, t)\| = O(1) \Leftrightarrow n^r \hat{f}(n) \lg |n| = O(1).$$

This completes the proof of Theorem 3.1. □

Corollary 3.1. *In the case of* $\alpha = 0$ *and* $r = 0$, *Theorem* 3.1 *is an extension of Theorem* 2.1 *in* [5].

Corollary 3.2. *Assume that* $\alpha \geq 0$, $r \in \{0, 1, \dots [\alpha]\}$, *and* C_n *is a complex zero sequence that satisfies* $\sum_{n=1}^{\infty} |\Delta(C_{-n} - C_n)| n^r \lg n < \infty$. *Let* $\rho_n > 0$ *such that* $\sum_{n=1}^{\infty} n^{\alpha}/n\rho_n < \infty$, $1/(n^{1-\beta}\rho_n) \downarrow 0$, *for some* $\beta > 0$, *and* $n^{p(r-\alpha)-1} \sum_{k=1}^{n} (k\rho_k)^p |\Delta C_k|^p = O(1)$. *Then the result of Theorem* 3.1 *is valid.*

Let

$$\begin{aligned} f_n(x) &= (a_0 - a_{n+1})/2 + \sum_{k=1}^{n} (a_k - a_{n+1}) \cos kx, \; S_n(x) \\ &= a_0/2 + \sum_{k=1}^{n} a_k \cos kx, \; a_n \\ &= O(1). \end{aligned}$$

If $S_n(x)$ is convergent, then its limit is denoted by $f(x)$. It is obvious that $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} S_n(x) = f(x)$.

Theorem 3.2. *Let* $a_n \in S_{p\alpha}$, $\alpha \geq 0$, $r \in \{0, 1, \dots, [\alpha]\}$. *Then*

(i) $\|f_n^{(r)}(x) - f^{(r)}(x)\| = O(n^{r-\alpha})$,

(ii) $\|S_n^{(r)}(x) - f^{(r)}(x)\| = O(n^{r-\alpha}) \Leftrightarrow a_n \lg n = O(n^{-\alpha})$.

Proof. It can be shown from $a_n \in S_{p\alpha}$ and from $A_k k^{\alpha} \leq 1/k$ that $\sum_{k=1}^{\infty} k^r |\Delta a_k| < \infty$. Indeed

$$\begin{aligned} \sum_{k=1}^{n} A_k k^r \frac{|\Delta a_k|}{A_k} &\leq \left(\sum_{k=1}^{n} k^{-q}\right)^{1/q} \left(\sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p}\right)^{1/p} \\ &= O(n^{-(1-1/q)+1/p}) = O(1). \end{aligned}$$

From Lemma 2.3 we know that $\sum_{k=0}^{\infty} D_k^{(r)}(x)\Delta a_k$ is uniformly convergent on any compact subset of $(0, \pi)$; $f(x) = \sum_{k=0}^{\infty} D_k(x)\Delta a_k$ implies $f^{(r)}(x) = \sum_{k=0}^{\infty} D_k^{(r)}(x)\Delta a_k$. So

$$\begin{aligned}\|f_n^{(r)}(x) - f^{(r)}(x)\| &= \left\| \sum_{k=n+1}^{\infty} D_k^{(r)}(x)\Delta a_k \right\| \\ &\leq 2\left\{ \sum_{k=n}^{\infty} |\Delta A_k| \int_0^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_k}{A_j} D_j^{(r)}(x) \right| dx + A_n \int_0^{\pi} \left| \sum_{j=1}^{n} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx \right\} \\ &= 2(J_1 + J_2).\end{aligned}$$

It follows from Lemma 2.4 and the estimates of I_{12} in Theorem 3.1 that $J_1 = o(n^{r-\alpha})$, $a_k \in S_{p\alpha}$. Similarly, $J_2 = o(n^{r-\alpha})$ holds.

From

$$S_n^{(r)}(x) - f^{(r)}(x) - a_{n+1} D_n^{(r)}(x) = f_n^{(r)}(x) - f^{(r)}(x),$$

$$|\|S_n^{(r)}(x) - f^{(r)}(x)\| - a_{n+1}\|D_n^{(r)}(x)\|| \leq \|f_n^{(r)}(x) - f^{(r)}(x)\|,$$

and Lemma 2.4, (ii) can be proved.

Corollary 3.3. *When* $r = 0$, $\alpha = 0$, *Theorem* 3.2 *becomes the extension of corresponding theorems in* [2].

Corollary 3.4. *If* $a_n \in S_\alpha \subset S_{p\alpha} (\alpha \geq 0)$, *Theorem* 3.2 *reduces to Theorem* 1 *in* [6].

References

1. N. Singh and K. M. Sharma, *Convergence of certain cosine sums in a metric space- L*, Proc. Amer. Math. Soc. **72** (1978), 117–120.
2. ——, *Convergence of a trigonometric series in the metric space L*, Arabian J. Sci. Engrg. **4** (1979), 137–140.
3. O. Szasz, *Quasi-monotone series*, Amer. J. Math. **70** (1948), 203–206.
4. J. W. Garrett, C. S. Ress and Č. V. Stanojevič, *Convergence of certain cosine sums in metric space- L*, Proc. Amer. Math. Soc. **72** (1978), 534–537.
5. Č. V. Stanojevič and V. B. Stanojevič, *Generalizations of Sidon-Telyakovskii Theorem*, Proc. Amer. Math. Soc. **101** (1987), 679–684.

6. Sheng Shuyun and Yang Yiqun, *The L^1 convergency of a type of trigonometric series*, Chinese J. Math. **4** (1985), 503–508.

7. S. A. Telyakovskii, *On a sufficient condition of sidon for the integrability of trigonometric series*, Math. Notes **14** (1973), 742–748.

Department of Mathematics, Hangzhou University, Hangzhou, Zhejiang, People's Republic of China