ON HYPERSPACES OF POLYHEDRA

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Abstract. Let $Q = [-1, 1]^\omega$ be the Hilbert cube and

$$Q_f = \{(x_i) \in Q | x_i = 0 \text{ except for finitely many } i\}.$$  

For a compact connected polyhedron $X$ with $\dim X > 0$, the hyperspaces of (nonempty) subcompacta, subcontinua, and compact subpolyhedra of $X$ are denoted by $2^X$, $C(X)$, and $\text{Pol}(X)$, respectively. And let $C^{\text{Pol}}(X) = C(X) \cap \text{Pol}(X)$. It is shown that the pair $(2^X, \text{Pol}(X))$ is homeomorphic to $(Q, Q_f)$. In case $X$ has no free arc, it is also proved that the pair $(C(X), C^{\text{Pol}}(X))$ is homeomorphic to $(Q, Q_f)$.

0. Introduction

The hyperspace $2^X$ of nonempty subcompacta of a nondegenerate Peano continuum $X$ is homeomorphic ($\cong$) to the Hilbert cube $Q = [-1, 1]^\omega$ and if $X$ has no free arc then the hyperspace $C(X)$ of nonempty subcontinua of $X$ is also homeomorphic to $Q$ [CSJ] (cf. [To]). All hyperspaces in this paper are topologized by the Hausdorff metric. The subset $s = (-1, 1)^\omega$ of $Q$ is called the pseudo-interior of $Q$ and $Q \setminus s$ the pseudo-boundary of $Q$. Let $\Sigma = \{(x_i) \in s | \sup |x_i| < 1\}$. Then $(Q, \Sigma) \cong (Q, Q \setminus s)$ and these sets $\Sigma$ and $Q \setminus s$ are topologically characterized as cap sets for $Q$ [An]. The hyperspace $2^X$ has several cap sets which naturally appear as hyperspaces [Mi1] (cf. [Mi2] and [CM]). Let us consider the following subsets of $Q$:

$$\sigma = \{(x_i) \in s | x_i = 0 \text{ except for finitely many } i\}$$

and

$$Q_f = \{(x_i) \in Q | x_i = 0 \text{ except for finitely many } i\}.$$  

Then $(Q, \sigma) \cong (Q, Q_f)$ and these sets $\sigma$ and $Q_f$ are topologically characterized as fd-cap sets for $Q$ [An]. If $X$ is a countable union of finite-dimensional (abbreviated fd) compacta, the hyperspace $\mathcal{F}(X)$ of nonempty finite sets in

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is an fd-cap set for \(2^X\), that is, \((2^X, \mathcal{F}(X)) \cong (Q, \sigma)\) [CN]. It seems that a "natural" fd-cap set (or cap set) for \(C(X)\) is unknown. The purpose of this paper is to obtain a natural fd-cap set for \(C(X)\) in case \(X\) is a polyhedron. For fd-cap (or cap) sets, we refer the reader to [Ch].

For a compact connected polyhedron \(X\), the hyperspace of nonempty compact subpolyhedra of \(X\) is denoted by \(\text{Pol}(X)\). And let \(C_{\text{Pol}}(X) = C(X) \cap \text{Pol}(X)\), the hyperspace of nonempty compact connected subpolyhedra of \(X\). The following theorem is our main result.

**Main Theorem.** Let \(X\) be a compact connected polyhedron with \(\dim X > 0\). Then \(\text{Pol}(X)\) is an fd-cap set for \(2^X\), that is, \((2^X, \text{Pol}(X)) \cong (Q, \sigma)\). In case \(X\) has no free arcs, \(C_{\text{Pol}}(X)\) is an fd-cap set for \(C(X)\), that is, \((C(X), C_{\text{Pol}}(X)) \cong (Q, \sigma)\).

### 1. Preliminaries

Let \(M = (M, d)\) be a metric space. For \(x \in M\) and \(A \subset M\), we denote \(d(x, A) = \inf\{d(x, a) | a \in A\}\). The diameter of \(A \subset M\) is denoted by \(\text{diam} A\). For a collection \(\mathcal{A}\) of subsets of \(M\), we denote \(\text{mesh} \mathcal{A} = \sup\{\text{diam} A | A \in \mathcal{A}\}\). For maps \(f, g : Y \to M\) from a space \(Y\) to \(M\), let \(d(f, g) = \sup\{d(f(x), g(x)) | x \in Y\}\). A closed set \(A \subset M\) is called a Z-set if for each map \(f : Q \to M\) and \(\varepsilon > 0\), there is a map \(g : Q \to M \setminus A\) with \(d(f, g) < \varepsilon\). A countable union of Z-sets in \(M\) is called a \(Z_\sigma\)-set. An fd-cap set (or a cap set) for \(M\) is a set \(A = \bigcup_{i \in N} A_i \subset M\), where \(A_i \subset A_{i+1}\), are fd compact Z-sets (or compact Z-sets) in \(M\) and the following condition is satisfied:

\[
\text{For each fd compactum (or compactum) } A \subset M, i \in N \text{ and } \varepsilon > 0, \text{ there is an embedding } h : A \to N_j \text{ of } A \text{ into some } N_j (\cap N_i) \text{ such that } h|A \cap N_i = \text{id} \text{ and } d(h, \text{id}) < \varepsilon.
\]

The following is due to R. D. Anderson [An] (cf. [Ch]).

1.1. **Lemma.** In order that \((M, N) \cong (Q, \sigma)\) (or \((M, N) \cong (Q, \Sigma)\)), it is necessary and sufficient that \(M \cong Q\) and \(N\) is an fd-cap set (or a cap set) for \(M\). □

Throughout the paper, let \(X = |K|\) denote a compact connected polyhedron with \(\dim X > 0\) and \(K\) a triangulation. We consider \(X\) as a (rectilinear) subpolyhedron of some \(\mathbb{R}^n\). By identifying \(x \in X\) and \(\{x\} \in 2^X\), we consider \(X \subset 2^X\). Do not confuse \(X \subset 2^X\) with \(X \subset 2^X\). Let \(\rho\) denote the path length metric on \(X\) defined by the metric inherited from Euclidean metric on \(\mathbb{R}^n\) (cf. [CN, Lemma 4.3]). Then \(\rho\) is convex in the sense of Menger [Me] and Euclidean on each simplex of \(K\). By \(\rho_H\), we denote the Hausdorff metric on \(2^X\) defined by

\[
\rho_H(A, B) = \max\{\sup\{\rho(x, A) | x \in B\}, \sup\{\rho(x, B) | x \in A\}\}.
\]
The barycenter, interior, and boundary of $\tau \in K$ are denoted by $\bar{\tau}$, $\mathring{\tau}$, and $\partial\tau$ and the star and link of $\tau \in K$ are denoted by $\text{St}(\tau, K)$ and $\text{Lk}(\tau, K)$, respectively. For each $v \in K^0$, let $O(v, K) = |\text{St}(v, K)| \setminus |\text{Lk}(v, K)|$. Then \(\{O(v, K)\mid v \in K^0\}\) is an open cover of $X$. The $n$-skeleton and the $n$th barycentric subdivision of $K$ are denoted by $K^n$ and $\text{Sd}^n K$, respectively. We write $\text{Sd}^1 K = \text{Sd} K$.

2. The hyperspace $\text{Pol}(X)$

In this section, we prove the first half of the Main Theorem, that is, $(2^X, \text{Pol}(X)) \cong (Q, \sigma)$. For each $n \in \mathbb{N}$, let $C_n(X)$ denote the subspace of $2^X$ consisting of all compacta with at most $n$ components. Then each $C_n(X)$ is a Z-set in $2^X$. In fact, for each $0 < \varepsilon < \text{diam} X$, we define $\kappa : 2^X \to 2^X$ by $\kappa(A) = \{x \in X : \rho(x, A) \leq \varepsilon\}$. Then $\kappa$ is continuous [Na, Corollary 3.4]. Since $\mathcal{T}(X)$ is an $f$-cap set for $2^X$, there is a map $\varphi : 2^X \to \mathcal{T}(X)$ with $d(\varphi, \text{id}) < \varepsilon/n$. As easily observed, each $\varphi \kappa(A)$ has at least $n+1$ components. Thus we have a map $\varphi \kappa : 2^X \to 2^X \setminus C_n(X)$ with $d(\varphi \kappa, \text{id}) < (n+1) \cdot \varepsilon/n$.

Now let $C_\omega(X) = \bigcup_{n \in \mathbb{N}} C_n(X)$ be the subset of $2^X$ consisting of all compacta with finitely many components. Then $C_\omega(X)$ is a $Z_\sigma$-set in $2^X$ and $\mathcal{T}(X) \subset \text{Pol}(X) \subset C_\omega(X)$. The first half of the theorem follows from [Ch, Lemma 4.2] and the following lemma.

2.1. Lemma. $\text{Pol}(X)$ is $\sigma$-fd-compact, that is, a countable union of fd compacta.

Proof. Let $Y$ be the convex hull of $X$ in $\mathbb{R}^n$. Then $\text{Pol}(X)$ is closed in $\text{Pol}(Y)$. If $\text{Pol}(Y)$ is $\sigma$-fd-compact, so is $\text{Pol}(X)$. Thus we may assume that $X$ is a compact convex set in $\mathbb{R}^n$. A triangulation of $P \in \text{Pol}(X)$ is said to be minimal if the number of vertices is minimal among all triangulation of $P$. Let $\mathcal{F}$ be the set of all finite subcomplexes of the countable infinite full complex $\Delta^\infty$. For each $L \in \mathcal{F}$, let $\text{Pol}_L(X)$ be the subset of $\text{Pol}(X)$ consisting of all polyhedra with a minimal triangulation which is simplicially isomorphic to $L$. It is straightforward to show that $\text{Pol}_L(X)$ is locally homeomorphic to $F_n(X) \setminus F_{n-1}(X)$, where $n$ is a number of vertices of $L$ and $F_n(X) = \{A \in 2^X : \text{card} A \leq n\}$. Since $F_n(X) \setminus F_{n-1}(X)$ is $\sigma$-fd-compact (cf. the proof of [CN, Lemma 3.1]), so is $\text{Pol}_L(X)$. Therefore $\text{Pol}(X) = \bigcup_{L \in \mathcal{F}} \text{Pol}_L(X)$ is also $\sigma$-fd-compact since $\mathcal{F}$ is countable. \(\square\)

3. The hyperspace $C^{\text{Pol}}(X)$

In this section, we prove the second half of the Main Theorem, that is, $(C(X), C^{\text{Pol}}(X)) \cong (Q, \sigma)$ if $X$ has no free arc. First we prove the following lemma in which $X$ may have a free arc.

3.1. Lemma. For each $\varepsilon > 0$ there exists a $\delta > 0$ such that each $\mathcal{A} \subset \text{Pol}(X)$ with $\text{diam} \mathcal{A} < \delta$ is contractible in a set $\mathcal{B} \subset \text{Pol}(X)$ such that $\text{diam} \mathcal{B} < \varepsilon$. 

each \( B \in \mathcal{B} \) contains some \( A \in \mathcal{A} \) and each component of \( B \) meets \( A \), hence \( \mathcal{B} \subset C^{Pol}(X) \) if \( \mathcal{A} \subset C^{Pol}(X) \).

Proof. Subdividing \( K \), we may assume that \( \text{mesh} K < \varepsilon/4 \). Choose \( \delta > 0 \) so that the closed \( \delta \)-neighborhood of each \( \text{St}(v, SdK) \) in \( K \) is contained in \( O(v, K) \). For each \( \mathcal{A} \subset Pol(X) \) and \( \text{diam} \mathcal{A} < \delta \), take an \( A_0 \in \mathcal{A} \) and let \( \{v_1, \ldots, v_n\} = \{v \in K^0 | A_0 \cap \text{St}(v, SdK) \neq \emptyset \} \) and \( V_i \) be the \( \delta \)-neighborhood of \( \text{St}(v_i, SdK) \) in \( K \). Then it follows that each \( A \in \mathcal{A} \) is contained in \( \bigcup_{i=1}^n V_i \) and meets each \( V_i \) because for each \( a \in A \) there is an \( a_0 \in A_0 \) with \( \rho(a, a_0) < \delta \) and conversely for each \( a_0 \in A_0 \) there is an \( a \in A \) with \( \rho(a, a_0) < \delta \).

We observe that if \( A \subset \bigcup_{i=1}^n \text{St}(v_i, K) \) and \( A \) meets each \( \text{St}(v_i, K) \), then \( \rho_H(A, A_0) < \varepsilon/2 \). In fact, each \( a \in A \) is contained in some \( \text{St}(v_i, K) \). Then \( \rho(a, v_i) \leq \text{mesh} K < \varepsilon/4 \). Since \( A_0 \cap \text{St}(v_i, SdK) \neq \emptyset \), we have an \( a_0 \in A_0 \) such that \( \rho(a_0, v_i) < \varepsilon/4 \), hence \( \rho(a, a_0) < \varepsilon/2 \). Conversely, each \( a_0 \in A_0 \) is contained in some \( \text{St}(v_i, SdK) \). Then \( \rho(a_0, v_i) < \varepsilon/4 \). Since \( A \cap \text{St}(v_i, K) \neq \emptyset \), we have an \( a \in A \) such that \( \rho(a, v_i) < \varepsilon/4 \); hence \( \rho(a, a_0) < \varepsilon/2 \). Therefore \( \rho_H(A, A_0) < \varepsilon/2 \).

Now we define a homotopy \( \varphi : \mathcal{A} \times I \to Pol(X) \) as follows:

\[
\varphi(A, t) = \bigcup_{i=1}^n \{sk_i(a) \cdot v_i + (1-s)k_i(a) \cdot a | a \in A \cap \text{St}(v_i, K) \}, \quad s \in [0, t] \}
\]

where \( k_i : \text{St}(v_i, K) \to I \) is a map with \( k_i(\text{cl} V_i) = 1 \) and \( k_i(\text{Lk}(v_i, K)) = 0 \). Then \( \varphi_0 = \text{id} \) and \( A \cup \{v_1, \ldots, v_n\} \subset \varphi_1(A) \) for each \( A \in \mathcal{A} \). By the fact observed above, \( \rho_H(\varphi_1(A), A_0) < \varepsilon/2 \) for each \( (A, t) \in \mathcal{A} \times I \). And we define a homotopy \( \psi : \mathcal{A} \times I \to Pol(X) \) as follows:

\[
\psi(A, t) = \varphi_1(A) \cup \bigcup_{i=1}^n \{(1-s) \cdot v_i + s \cdot x | x \in \text{St}(v_i, K) \}, \quad s \in [0, t] \}
\]

Then \( \psi_0 = \varphi_1 \) and \( \psi_1(A) = \bigcup_{i=1}^n \text{St}(v_i, K) \) for each \( A \in \mathcal{A} \). Again by the fact observed above, \( \rho_H(\psi_1(A), A_0) < \varepsilon/2 \) for each \( (A, t) \in \mathcal{A} \times I \). Then \( \mathcal{A} \) is contractible in the set \( \mathcal{B} = \varphi(\mathcal{A} \times I) \cup \psi(\mathcal{A} \times I) \subset Pol(X) \) with \( \text{diam} \mathcal{B} < \varepsilon \).

Observe that each \( B \in \mathcal{B} \) contains some \( A \in \mathcal{A} \) and each component of \( B \) meets \( A \). □

Hereafter we assume that \( X \) has no free arc.

3.2. Lemma. \( C^{Pol}(X) \) is a \( \mathbb{Z}_\sigma \)-set in \( C(X) \).

Proof. By Lemma 2.1, \( C^{Pol}(X) \) is \( \sigma \)-compact, so \( F_\sigma \) in \( C(X) \). Then it suffices to show that for each \( \varepsilon > 0 \), there is a map \( \varphi : C(X) \to C(X) \setminus Pol(X) \) with \( \rho_H(\varphi, \text{id}) < \varepsilon \). Subdividing \( K \), we may assume that \( \text{mesh} K < \varepsilon/2 \). Let \( \{\tau_i | i = 1, \ldots, n\} \) be the set of principal (= maximal dimensional) simplexes of \( K \). Then \( X = \bigcup_{i=1}^n \tau_i \). Choose Euclidean balls \( B_i \) in \( \dot{\tau}_i = \text{int} \tau_i \) such that \( \dim B_i = \dim \tau_i \geq 2 \) and set \( Y = X \setminus \bigcup_{i=1}^n \dot{B}_i \). Since \( C(\partial B_i) \) is an AR, we
have a map $f_i : B_i \to C(\partial B_i)$ such that $f_i|\partial B_i = \text{id}$. Let $f : X \to C(Y)$ be the map defined by $f|Y = \text{id}$ and $f|B_i = f_i$. Then $f$ induces the map $\theta : C(X) \to C(Y) \subset C(X)$ defined by $\theta(A) = \bigcup \{f(a)|a \in A\}$ (cf. [Ke]). It is easy to see that $\rho_H(\theta, \text{id}) < \varepsilon/2$. If $A$ contains some $B_i$, then $\theta(A)$ contains $\partial B_i$ but misses $B_i$ hence any point of $\partial B_i$ has no cone neighborhood in $\theta(A)$ (because $\text{dim} B_i \geq 2$), that is, $\theta(A)$ is not a polyhedron (cf. [RS]). Let $\kappa : C(X) \to C(X)$ be the map defined by $\kappa(A) = \{x \in X|\rho(x, A) \leq \varepsilon/2\}$. Clearly $\rho_H(\kappa, \text{id}) \leq \varepsilon/2$ and each $\kappa(A)$ contains some $B_i$. Thus we have the map $\varphi = \theta \circ \kappa : C(X) \to C(X) \setminus \text{Pol}(X)$ with $\rho_H(\varphi, \text{id}) < \varepsilon$. 

For each $n \in \mathbb{N}$, let $Y_n = |(\text{Sd}^{2n} K)^1|$ and

$$P_n = \{A \in C(Y_n)|\exists \tau \in \text{Sd}^{2n} K \text{ s.t. dim} \tau = 1 \text{ and } \tau \subset A\}.$$ 

Then each $P_n$ is closed in $C(Y_n)$, and is therefore $\text{fd}$ compact.

3.3. **Lemma.** For each $n \in \mathbb{N}$, there exists an embedding $\xi : P_n \times I \to P_{n+1}$.

**Proof.** We use the path length metric $\rho'$ on $Y_{n+1}$. One should note that $\rho'$ is not the restriction of $\rho$. Let

$$\beta = \min\{\rho'(v, v')|v \neq v' \in (\text{Sd}^{2n+2} K)^0\}$$

$$\leq \frac{1}{4} \cdot \min\{\rho'(v, v')|v \neq v' \in (\text{Sd}^{2n} K)^0\}.$$ 

Then for points $x$ and $x'$ in any connected subgraph $Z$ of $Y_{n+1}$, if $\rho'(x, x') < \beta$ then $\rho'(x, x')$ is the distance between $x$ and $x'$ with respect to the path length metric on $Z$. Let $\kappa : P_n \times I \to P_n$ be the map defined by $\kappa(A, t) = \{x \in Y_n|\rho'(x, A) \leq t\beta\}$. For each $(A, t) \in P_n \times I$ and $\tau \in \text{Sd}^n K$ with dim $\tau = 1$, if $t\beta \geq \frac{1}{2} \cdot \text{diam}(\tau \setminus A)$ then $\tau \cap A \neq \emptyset$ and $\tau \in \kappa(A, t)$, where $\text{diam} \emptyset = 0$. Then we can define a map $\eta : P_n \times I \to P_{n+1}$ as follows:

$$\eta(A, t) = \kappa(A, t) \cup \{x \in |\text{St}(\hat{\tau}, \text{Sd}^{2n+1} K)^1|\tau \in \text{Sd}^{2n} K,$$

$$\text{dim} \tau = 1, \rho'(x, \hat{\tau}) \leq t\beta - \frac{1}{2} \cdot \text{diam}(\tau \setminus A)\}.$$ 

And we define $\zeta : P_n \times I \to P_{n+1}$ as follows:

$$\zeta(A, t) = A \cup \{x \in |\text{St}(\hat{\tau}, \text{Sd}^{2n+2} K)^1|\tau \leq \tau \in (\text{Sd}^{2n} K)^1,$$

$$\tau \setminus A \neq \emptyset, 0 < \rho'(x, \tau) \leq t \cdot \delta(A, \tau, v)\},$$

where

$$\delta(A, \tau, v) = \frac{\rho'(v, \tau \setminus A) \cdot \text{diam}(\tau \setminus A) \cdot \beta}{(\text{diam} \tau)^2} < \beta.$$ 

It is straightforward to see that $\zeta$ is continuous. The desired embedding $\xi : P_n \times I \to P_{n+1}$ can be defined by $\xi(A, t) = \eta(A, t) \cup \zeta(A, t)$. In fact, each $A \in P_n$ contains some 1-simplex $\tau \in \text{Sd}^n K$ and then

$$\rho'(\hat{\tau}, |\text{St}(\hat{\tau}, \text{Sd}^{2n+1} K)^1| \setminus \xi(A, t)) = t\beta.$$
Hence $t \neq t'$ implies $\xi(A, t) \neq \xi(A', t')$ for any $A, A' \in P_n$. If $A \neq A' \in P_n$ then $\tau \cap A \neq \tau \cap A'$ for some 1-simplex $\tau \in \text{Sd}^n K$. In case $\text{diam}(\tau \setminus A) \neq \text{diam}(\tau \setminus A')$, $\eta(A, t) \neq \eta(A', t)$. In case $\text{diam}(\tau \setminus A) = \text{diam}(\tau \setminus A') > 0$, $\delta(A, \tau, v) \neq \delta(A', \tau, v)$ for at least one vertex $v$ of $\tau$, so $\xi(A, t) \neq \xi(A', t)$. Thus $A \neq A'$ implies $\xi(A, t) \neq \xi(A', t)$ for each $t \in [0, 1]$. Therefore $\xi$ is an embedding. By definition, $\xi(A, 0) = A \subset Y_n$ for each $A \in P_n$. \hfill $\square$

3.4. Lemma. $\bigcup_{n \in \mathbb{N}} P_n$ is dense in $C(X)$.

Proof. For each $A \in C(X)$ and $\varepsilon > 0$, choose $n \in \mathbb{N}$ so that $\text{mesh}(\text{Sd}^n K) < \varepsilon$ and set $B = Y_n \cap \bigcup\{\tau \in \text{Sd}^n K | A \cap \tau \neq \emptyset\} \subset P_n$. Then $\rho_H(A, B) < \varepsilon$. Therefore $\bigcup_{n \in \mathbb{N}} P_n$ is dense in $C(X)$. \hfill $\square$

3.5. Lemma. $\bigcup_{n \in \mathbb{N}} P_n$ is an fd-cap set for $C(X)$.

Proof. By Lemma 3.2, each $P_n$ is an fd compact Z-set in $C(X)$. Let $A \supset B$ be fd compacta, $f : A \to C(X)$ a map such that $f|B : B \to P_n$ is an embedding into some $P_n$, and let $\varepsilon > 0$. Note that in Lemma 3.1 $B \subset P_m$ if $A \subset P_m$. Similar to [GH, Lemma 2], by using Lemmas 3.4 and 3.1, we can construct a map $g : A \to P_m$ of $A$ into some $P_m \supset P_n$ such that $\rho_H(g, f) < \varepsilon/2$ and $g|B = f|B$. As in the proof of [CN, Lemma 4.6], by using Lemma 3.3 we can replace $g$ by an embedding $h : A \to P_l$ of $A$ into some $P_l \supset P_m$ such that $h|B = g|B = f|B$ and $\rho_H(h, g) < \varepsilon/2$ so $\rho_H(h, f) < \varepsilon$. Therefore $\bigcup_{n \in \mathbb{N}} P_n$ is an fd-cap set for $C(X)$. \hfill $\square$

Now we complete the proof of the second part of the Main Theorem. Since $\bigcup_{n \in \mathbb{N}} P_n \subset C_{\text{Pol}}(X)$, the result follows from Lemmas 3.5, 2.1, and 3.2, and [Ch, Lemma 4.2]. \hfill $\square$

4. Star hyperspaces

Recall that $K$ is a finite connected simplicial complex with $|K| = X$. In [CS], Curtis and Schori introduced the star hyperspace $2^X_{\text{sst}}$ of $K$ by restricting the size of compacta in $2^X$ as follows. For each $v \in K^0$, let

$$\text{sst}(v) = \bigcup\{\tau \in \text{Sd}^2 K | \text{St}(v, \text{Sd} K) \cap \tau \neq \emptyset\}.$$ 

We define the following star hyperspaces.

\[
\begin{align*}
2^K_{\text{sst}} &= \{A \in 2^X | A \subset \text{sst}(v) \text{ for some } v \in K^0\}; \\
C_{\text{sst}}(K) &= C(X) \cap 2^K_{\text{sst}}; \\
C_{\omega_{\text{sst}}}^\omega(K) &= C_{\omega}(X) \cap 2^K_{\text{sst}}; \\
\text{Pol}_{\text{sst}}(K) &= \text{Pol}(X) \cap 2^K_{\text{sst}}; \\
C_{\text{Pol}}_{\text{sst}}(K) &= C_{\text{Pol}}(X) \cap 2^K_{\text{sst}}; \\
\mathcal{F}_{\text{sst}}(K) &= \mathcal{F}(X) \cap 2^K_{\text{sst}}.
\end{align*}
\]

In [CS], Curtis and Schori proved that $2^K_{\text{sst}} \cong X \times Q$ and that $C_{\text{sst}}(K) \cong X \times Q$ if $X$ has no free arcs. This is generalized as follows.
4.1. **Theorem.** For any finite connected simplicial complex $K$, 

$$(S^*_s(K), S^*_s(K)) \cong (|K| \times Q, |K| \times \Sigma),$$

$$(\text{Pol}^*_s(K), \text{Pol}^*_s(K)) \cong (|K| \times Q, |K| \times \Sigma).$$

*In case $K$ has no principal 1-simplex,*

$$(\text{C}^*_s(K), \text{C}^*_s(K)) \cong (|K| \times Q, |K| \times \Sigma),$$

$$(\text{C}_{\omega s}^s(K), \text{C}_{\omega s}^s(K)) \cong (|K| \times Q, |K| \times \Sigma).$$

This theorem is valid for a locally finite simplicial complex $K$ without isolated vertices. To prove the theorem, we need the relative version of [CS2, Theorem 3.5]. A locally finite closed cover $\mathcal{D} = \{Q_k\}$ of a metric space $M$ is called a $Q$-decomposition if (i) each $Q_k$ is homeomorphic to $Q$; (ii) $\mathcal{D}$ contains all nonempty intersections of its members; and (iii) if $Q_j$ is a proper subset of $Q_k$, then $Q_j$ is a Z-set in $Q_k$. We say that $Q$-decompositions $\{Q_k\}$ and $\{Q'_k\}$ of spaces $M$ and $M'$ are isomorphic if they are order isomorphic with respect to the partial orderings given by set inclusion. It is straightforward to see the following version of [CS2, Theorem 3.5].

4.2. **Theorem.** Let $\{Q_i\}$ and $\{Q'_i\}$ be isomorphic $Q$-decompositions of spaces $M$ and $M'$, $N \subset M$ and $N' \subset M'$ such that each $Q_i \cap N$ and $Q'_i \cap N'$ are fd-cap sets (or cap sets) for $Q_i$ and $Q'_i$, respectively. Then there exists a homeomorphism of $M$ onto $M'$ taking each element of $\{Q_i\}$ onto the corresponding element of $\{Q'_i\}$ and $N$ onto $N'$.

**Proof of Theorem 4.1.** For $A_1, \ldots, A_m \in 2^X$, let

$$2^X(A_1, \ldots, A_m) = \{A \in 2^X | A \cap A_i \neq \emptyset \text{ for each } i\},$$

$$C(X; A_1, \ldots, A_m) = C(X) \cap 2^X(A_1, \ldots, A_m).$$

First, by [CN, Corollary 5.2], $\mathcal{F}(X; A_1, \ldots, A_m) = \mathcal{F}(X) \cap 2^X(A_1, \ldots, A_m)$ is an fd-cap set for $2^X(A_1, \ldots, A_m)$. As in §2, we can see that

$$C_{\omega}(X; A_1, \ldots, A_m) = C_{\omega}(X) \cap 2^X(A_1, \ldots, A_m)$$

is a $Z_{\sigma}$-set in $2^X(A_1, \ldots, A_m)$, so

$$\text{Pol}(X; A_1, \ldots, A_m) = \text{Pol}(X) \cap 2^X(A_1, \ldots, A_m)$$

is also an fd-cap set for $2^X(A_1, \ldots, A_m)$. Similar to [CS2, Theorem 4.3], the first half of the theorem follows from Theorem 4.2.

In the case where $X$ has no free arcs, $C_{\omega}(X; A_1, \ldots, A_m)$ is a cap set for $2^X(A_1, \ldots, A_m)$ by the result of [Mi1]. By choosing $B_i$ in the proof of Lemma 3.2 so that $B_i \cap A_j = \emptyset$ or $B_i \subsetneq A_j$ for each $j = 1, \ldots, m$, we can prove that

$$C_{\text{Pol}}^\Sigma(X; A_1, \ldots, A_m) = C_{\text{Pol}}^\Sigma(X) \cap 2^X(A_1, \ldots, A_m)$$
is a $Z_\sigma$-set for $C(X; A_1, \ldots, A_m)$. Now we assume that each $A_i$ is a polyhedron triangulated by a subcomplex of $Sd^2 K$. Then each $A_i$ meets each $Y_n$ defined in §3. For each $n \in \mathbb{N}$, let $P_n(A_1, \ldots, A_m) = P_n \cap 2^X(A_1, \ldots, A_m)$, where $P_n$ is defined in §3. Then we can prove, in a way similar to Lemma 3.5, that $\bigcup_{n \in \mathbb{N}} P_n(A_1, \ldots, A_m)$ is an fd-cap set for $C(X; A_1, \ldots, A_m)$. Since $\bigcup_{n \in \mathbb{N}} P_n(A_1, \ldots, A_m) \subset C_{Pol}(X; A_1, \ldots, A_m)$, $C_{Pol}(X; A_1, \ldots, A_m)$ is an fd-cap set for $C(X; A_1, \ldots, A_m)$. Therefore the second half also follows from Theorem 4.2. □

5. Remarks

In the case where $X$ is a finite graph, we have $Pol(X) = C_{\omega}(X)$ and $C_{Pol}(X) = C(X)$. The first assertion of the theorem has been shown in this case by Michael [Mi,2] (cf. [Mi,2]), and the assumption of the second statement is essential.

The second part of the Main Theorem can be generalized as follows.

5.1. Theorem. If $X$ is a compact connected polyhedron with no free arc, then for each $n \in \mathbb{N}$, $(C_n(X), C_n(X) \cap Pol(X)) \cong (Q, \sigma)$.

Proof. First note that in Lemma 3.1, $\mathcal{B} \subset C_n(X) \cap Pol(X)$ if $\mathcal{A} \subset C_n(X) \cap Pol(X)$. In the proof of Lemma 3.2, the map $\varphi$ can be defined on $2^X$. In this case, we have $\varphi(C_n(X)) \subset C_n(X)$ for each $n \in \mathbb{N}$. Hence it follows that $C_n(X) \cap Pol(X)$ is also a $Z_\sigma$-set in $C_n(X)$ for each $n \in \mathbb{N}$ if $X$ has no free arc. As is easily seen, $C_n(X) \cap Pol(X)$ is dense in $C_n(X)$. For each $m, n \in \mathbb{N}$, let $Y_m = |(Sd^m K)^1|$ and $P_{n,m} = \{ A \in C_n(Y_m) | A \text{ contains some } \tau \in (Sd^m K)^1 \}$. As in §3, we can prove that $\bigcup_{m,n \in \mathbb{N}} P_{n,m}$ is an fd-cap set for $C_n(X)$. For each $n \in \mathbb{N}$, $C_n(X) \cong Q$ and $\bigcup_{m,n \in \mathbb{N}} P_{n,m} \subset C_n(X) \cap Pol(X)$. (E.g., $C_n(X) \cong Q$ follows from [Cu, Corollary 5.1] since it is as easy to show as in Lemma 3.2 that $\{X\}$ is a $Z$-set in $C_n(X)$.) Thus we have the result similar to the second part of the Main Theorem. □

From Theorem 5.1, it is natural to conjecture as follows.

5.2. Conjecture. For a compact connected polyhedron $X$ with no free arc, $(2^X, C_{\omega}(X), Pol(X)) \cong (Q, \Sigma, \sigma)$.

Applying the characterization of the triple $(Q, \Sigma, \sigma)$ in [SW], this conjecture is reduced to show that each $C_n(X)$ is a $Z$-set in some $C_m(X)$ (cf. [Sa]). And related to the result of §4, it is also natural to conjecture as follows.

5.3. Conjecture. For a (locally) finite simplicial complex $K$ with no principal 1-simplexes,

$$(2^K, C_{\omega ss}(K), Pol_{ss}(K)) \cong ([K] \times Q, [K] \times \Sigma, [K] \times \sigma).$$

One may also conjecture that $(2^X, C_{\omega}(X), \mathcal{F}(X)) \cong (Q, \Sigma, \sigma)$ since $\mathcal{F}(X) \subset C_{\omega}(X) \subset 2^X$, $(2^X, C_{\omega}(X)) \cong (Q, \Sigma)$, and $(2^X, \mathcal{F}(X)) \cong (Q, \sigma)$. However
this conjecture is false. In fact, \( C_\omega(X) \setminus \mathcal{F}(X) = \bigcup_{n \in \mathbb{N}} C_n(X) \setminus \mathcal{F}(X) \) and each \( C_n(X) \setminus \mathcal{F}(X) \) is \( \sigma \)-compact because \( \mathcal{F}(X) \cap C_n(X) = \mathcal{F}(X) \) is compact, so it is closed in \( C_n(X) \). Hence \( C_\omega(X) \setminus \mathcal{F}(X) \) is also \( \sigma \)-compact. On the other hand, \( \Sigma \setminus \sigma \) is not \( \sigma \)-compact because \( \Sigma \setminus \sigma \cong \Sigma \times \sigma \) by [SW, Theorem 3.1]. This observation is due to Jan van Mill. The author would like to thank Jan van Mill for his observation.

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