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ON HYPERSPACES OF POLYHEDRA

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ABSTRACT. Let $Q = [-1, 1]^\omega$ be the Hilbert cube and

$$Q_f = \{(x_i) \in Q | x_i = 0 \text{ except for finitely many } i\}.$$

For a compact connected polyhedron X with $\dim X > 0$, the hyperspaces of (nonempty) subcompacta, subcontinua, and compact subpolyhedra of X are denoted by 2^X, $C(X)$, and $\mathrm{Pol}(X)$, respectively. And let $C^{\mathrm{Pol}}(X) = C(X) \cap \mathrm{Pol}(X)$. It is shown that the pair $(2^X, \mathrm{Pol}(X))$ is homeomorphic to (Q, Q_f). In case X has no free arc, it is also proved that the pair $(C(X), C^{\mathrm{Pol}}(X))$ is homeomorphic to (Q, Q_f).

0. INTRODUCTION

The hyperspace 2^X of nonempty subcompacta of a nondegenerate Peano continuum X is homeomorphic ($\cong$) to the Hilbert cube $Q = [-1, 1]^\omega$ and if X has no free arc then the hyperspace $C(X)$ of nonempty subcontinua of X is also homeomorphic to Q [CS_1] (cf. [To]). All hyperspaces in this paper are topologized by the Hausdorff metric. The subset $s = (-1, 1)^\omega$ of Q is called the pseudo-interior of Q and $Q \setminus s$ the pseudo-boundary of Q. Let $\Sigma = \{(x_i) \in s | \sup |x_i| < 1\}$. Then $(Q, \Sigma) \cong (Q, Q \setminus s)$ and these sets Σ and $Q \setminus s$ are topologically characterized as cap sets for Q [An]. The hyperspace 2^X has several cap sets which naturally appear as hyperspaces [Mi_1] (cf. [Mi_2] and [CM]). Let us consider the following subsets of Q:

$$\sigma = \{(x_i) \in s | x_i = 0 \text{ except for finitely many } i\}$$

and

$$Q_f = \{(x_i) \in Q | x_i = 0 \text{ except for finitely many } i\}.$$

Then $(Q, \sigma) \cong (Q, Q_f)$ and these sets σ and Q_f are topologically characterized as fd-cap sets for Q [An]. If X is a countable union of finite-dimensional (abbreviated fd) compacta, the hyperspace $\mathscr{F}(X)$ of nonempty finite sets in

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X is an fd-cap set for 2^X, that is, $(2^X, \mathscr{F}(X)) \cong (Q, \sigma)$ [CN]. It seems that a "natural" fd-cap set (or cap set) for $C(X)$ is unknown. The purpose of this paper is to obtain a natural fd-cap set for $C(X)$ in case X is a polyhedron. For fd-cap (or cap) sets, we refer the reader to [Ch].

For a compact connected polyhedron X, the hyperspace of nonempty compact subpolyhedra of X is denoted by $\mathrm{Pol}(X)$. And let $C^{\mathrm{Pol}}(X) = C(X) \cap \mathrm{Pol}(X)$, the hyperspace of nonempty compact connected subpolyhedra of X. The following theorem is our main result.

Main Theorem. *Let X be a compact connected polyhedron with* $\dim X > 0$. *Then* $\mathrm{Pol}(X)$ *is an* fd-cap *set for* 2^X, *that is,* $(2^X, \mathrm{Pol}(X)) \cong (Q, \sigma)$. *In case X has no free arcs,* $C^{\mathrm{Pol}}(X)$ *is an* fd-cap *set for* $C(X)$, *that is,* $(C(X), C^{\mathrm{Pol}}(X)) \cong (Q, \sigma)$.

1. Preliminaries

Let $M = (M, d)$ be a metric space. For $x \in M$ and $A \subset M$, we denote $d(x, A) = \inf\{d(x, a) | a \in A\}$. The diameter of $A \subset M$ is denoted by $\operatorname{diam} A$. For a collection $\mathscr{A}$ of subsets of M, we denote $\operatorname{mesh} \mathscr{A} = \sup\{\operatorname{diam} A | A \in \mathscr{A}\}$. For maps $f, g : Y \to M$ from a space Y to M, let $d(f, g) = \sup\{d(f(x), g(x)) | x \in Y\}$. A closed set A in M is called a *Z-set* if for each map $f : Q \to M$ and $\varepsilon > 0$, there is a map $g : Q \to M \setminus A$ with $d(f, g) < \varepsilon$. A countable union of Z-sets in M is called a *Z_σ-set.* An *fd-cap set* (or a *cap set*) for M is a set $N = \bigcup_{i \in \mathbf{N}} N_i \subset M$, where $N_1 \subset N_2 \subset \cdots$ are fd compact Z-sets (or compact Z-sets) in M and the following condition is satisfied:

(*) For each fd compactum (or compactum) $A \subset M$, $i \in N$ and $\varepsilon > 0$, there is an embedding $h : A \to N_j$ of A into some $N_j (\supset N_i)$ such that $h|A \cap N_i = \mathrm{id}$ and $d(h, \mathrm{id}) < \varepsilon$.

The following is due to R. D. Anderson [An] (cf. [Ch]).

1.1. **Lemma.** *In order that* $(M, N) \cong (Q, \sigma)$ *(or* $(M, N) \cong (Q, \Sigma)$*), it is necessary and sufficient that* $M \cong Q$ *and* N *is an* fd-cap *set (or a* cap *set) for* M. □

Throughout the paper, *let* $X = |K|$ *denote a compact connected polyhedron with* $\dim X > 0$ *and* K *a triangulation.* We consider X as a (rectilinear) subpolyhedron of some $\mathbf{R}^n$. By identifying $x \in X$ and $\{x\} \in 2^X$, we consider $X \subset 2^X$. Do not confuse $X \in 2^X$ with $X \subset 2^X$. Let ρ denote the path length metric on X defined by the metric inherited from Euclidean metric on $\mathbf{R}^n$ (cf. [CN, Lemma 4.3]). Then ρ is convex in the sense of Menger [Me] and Euclidean on each simplex of K. By ρ_H, we denote the Hausdorff metric on 2^X defined by

$$\rho_H(A, B) = \max\{\sup\{\rho(x, A) | x \in B\}, \sup\{\rho(x, B) | x \in A\}\}.$$

The barycenter, interior, and boundary of $\tau \in K$ are denoted by $\hat{\tau}$, $\dot{\tau}$, and $\partial\tau$ and the star and link of $\tau \in K$ are denoted by $\mathrm{St}(\tau, K)$ and $\mathrm{Lk}(\tau, K)$, respectively. For each $v \in K^0$, let $\mathrm{O}(v, K) = |\mathrm{St}(v, K)| \setminus |\mathrm{Lk}(v, K)|$. Then $\{\mathrm{O}(v, K) | v \in K^0\}$ is an open cover of X. The n-skeleton and the nth barycentric subdivision of K are denoted by K^n and $\mathrm{Sd}^n K$, respectively. We write $\mathrm{Sd}^1 K = \mathrm{Sd}\, K$.

2. The hyperspace $\mathrm{Pol}(X)$

In this section, we prove the first half of the Main Theorem, that is, $(2^X, \mathrm{Pol}(X)) \cong (Q, \sigma)$. For each $n \in \mathbf{N}$, let $C_n(X)$ denote the subspace of 2^X consisting of all compacta with at most n components. Then each $C_n(X)$ is a Z-set in 2^X. In fact, for each $0 < \varepsilon < \mathrm{diam}\, X$, we define $\kappa : 2^X \to 2^X$ by $\kappa(A) = \{x \in X | \rho(x, A) \leq \varepsilon\}$. Then κ is continuous [Na, Corollary 3.4]. Since $\mathscr{F}(X)$ is an fd-cap set for 2^X, there is a map $\varphi : 2^X \to \mathscr{F}(X)$ with $d(\varphi, \mathrm{id}) < \varepsilon/n$. As easily observed, each $\varphi\kappa(A)$ has at least $n+1$ components. Thus we have a map $\varphi\kappa : 2^X \to 2^X \setminus C_n(X)$ with $d(\varphi\kappa, \mathrm{id}) < (n+1) \cdot \varepsilon/n$. Now let $C_\omega(X) = \bigcup_{n \in \mathbf{N}} C_n(X)$ be the subset of 2^X consisting of all compacta with finitely many components. Then $C_\omega(X)$ is a Z_σ-set in 2^X and $\mathscr{F}(X) \subset \mathrm{Pol}(X) \subset C_\omega(X)$. The first half of the theorem follows from [Ch, Lemma 4.2] and the following lemma.

2.1. **Lemma.** $\mathrm{Pol}(X)$ *is* σ-fd-*compact, that is, a countable union of* fd *compacta.*

Proof. Let Y be the convex hull of X in $\mathbf{R}^n$. Then $\mathrm{Pol}(X)$ is closed in $\mathrm{Pol}(Y)$. If $\mathrm{Pol}(Y)$ is σ-fd-compact, so is $\mathrm{Pol}(X)$. Thus we may assume that X is a compact convex set in $\mathbf{R}^n$. A triangulation of $P \in \mathrm{Pol}(X)$ is said to be *minimal* if the number of vertices is minimal among all triangulation of P. Let $\mathscr{F}$ be the set of all finite subcomplexes of the countable infinite full complex Δ^∞. For each $L \in \mathscr{F}$, let $\mathrm{Pol}_L(X)$ be the subset of $\mathrm{Pol}(X)$ consisting of all polyhedra with a minimal triangulation which is simplicially isomorphic to L. It is straightforward to show that $\mathrm{Pol}_L(X)$ is locally homeomorphic to $F_n(X) \setminus F_{n-1}(X)$, where n is a number of vertices of L and $F_n(X) = \{A \in 2^X | \mathrm{card}\, A \leq n\}$. Since $F_n(X) \setminus F_{n-1}(X)$ is σ-fd-compact (cf. the proof of [CN, Lemma 3.1]), so is $\mathrm{Pol}_L(X)$. Therefore $\mathrm{Pol}(X) = \bigcup_{L \in \mathscr{F}} \mathrm{Pol}_L(X)$ is also σ-fd-compact since $\mathscr{F}$ is countable. □

3. The hyperspace $C^{\mathrm{Pol}}(X)$

In this section, we prove the second half of the Main Theorem, that is, $(C(X), C^{\mathrm{Pol}}(X)) \cong (Q, \sigma)$ if X has no free arc. First we prove the following lemma in which X may have a free arc.

3.1. **Lemma.** *For each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that each* $\mathscr{A} \subset \mathrm{Pol}(X)$ *with* $\mathrm{diam}\, \mathscr{A} < \delta$ *is contractible in a set* $\mathscr{B} \subset \mathrm{Pol}(X)$ *such that* $\mathrm{diam}\, \mathscr{B} < \varepsilon$,

each $B \in \mathscr{B}$ *contains some* $A \in \mathscr{A}$ *and each component of* B *meets* A, *hence* $\mathscr{B} \subset C^{\mathrm{Pol}}(X)$ *if* $\mathscr{A} \subset C^{\mathrm{Pol}}(X)$.

Proof. Subdividing K, we may assume that $\operatorname{mesh} K < \varepsilon/4$. Choose $\delta > 0$ so that the closed δ-neighborhood of each $|\mathrm{St}(v, \mathrm{Sd}\, K)|$ in $|K|$ is contained in $O(v, K)$. For each $\mathscr{A} \subset \mathrm{Pol}(X)$ and $\operatorname{diam} \mathscr{A} < \delta$, take an $A_0 \in \mathscr{A}$ and let $\{v_1, \dots, v_n\} = \{v \in K^0 | A_0 \cap |\mathrm{St}(v, \mathrm{Sd}\, K)| \neq \emptyset\}$ and V_i be the δ-neighborhood of $|\mathrm{St}(v_i, \mathrm{Sd}\, K)|$ in $|K|$. Then it follows that each $A \in \mathscr{A}$ is contained in $\bigcup_{i=1}^n V_i$ and meets each V_i because for each $a \in A$ there is an $a_0 \in A_0$ with $\rho(a, a_0) < \delta$ and conversely for each $a_0 \in A_0$ there is an $a \in A$ with $\rho(a, a_0) < \delta$.

We observe that if $A \subset \bigcup_{i=1}^n |\mathrm{St}(v_i, K)|$ and A meets each $|\mathrm{St}(v_i, K)|$, then $\rho_H(A, A_0) < \varepsilon/2$. In fact, each $a \in A$ is contained in some $|\mathrm{St}(v_i, K)|$. Then $\rho(a, v_i) \leq \operatorname{mesh} K < \varepsilon/4$. Since $A_0 \cap |\mathrm{St}(v_i, \mathrm{Sd}\, K)| \neq \emptyset$, we have an $a_0 \in A_0$ such that $\rho(a_0, v_i) < \varepsilon/4$, hence $\rho(a, a_0) < \varepsilon/2$. Conversely, each $a_0 \in A_0$ is contained in some $|\mathrm{St}(v_i, \mathrm{Sd}\, K)|$. Then $\rho(a_0, v_i) < \varepsilon/4$. Since $A \cap |\mathrm{St}(v_i, K)| \neq \emptyset$, we have an $a \in A$ such that $\rho(a, v_i) < \varepsilon/4$; hence $\rho(a, a_0) < \varepsilon/2$. Therefore $\rho_H(A, A_0) < \varepsilon/2$.

Now we define a homotopy $\varphi : \mathscr{A} \times I \to \mathrm{Pol}(X)$ as follows:

$$\varphi(A, t) = \bigcup_{i=1}^n \{sk_i(a) \cdot v_i + (1-s)k_i(a) \cdot a | a \in A \cap |\mathrm{St}(v_i, K)|, \ s \in [0, t]\}.$$

where $k_i : |\mathrm{St}(v_i, K)| \to I$ is a map with $k_i(\mathrm{cl}\, V_i) = 1$ and $k_i(|\mathrm{Lk}(v_i, K)|) = 0$. Then $\varphi_0 = \mathrm{id}$ and $A \cup \{v_1, \dots, v_n\} \subset \varphi_1(A)$ for each $A \in \mathscr{A}$. By the fact observed above, $\rho_H(\varphi_t(A), A_0) < \varepsilon/2$ for each $(A, t) \in \mathscr{A} \times I$. And we define a homotopy $\psi : \mathscr{A} \times I \to \mathrm{Pol}(X)$ as follows:

$$\psi(A, t) = \varphi_1(A) \cup \bigcup_{i=1}^n \{(1-s) \cdot v_i + s \cdot x | x \in |\mathrm{St}(v_i, K)|, \ s \in [0, t]\}.$$

Then $\psi_0 = \varphi_1$ and $\psi_1(A) = \bigcup_{i=1}^n |\mathrm{St}(v_i, K)|$ for each $A \in \mathscr{A}$. Again by the fact observed above, $\rho_H(\psi_t(A), A_0) < \varepsilon/2$ for each $(A, t) \in \mathscr{A} \times I$. Then $\mathscr{A}$ is contractible in the set $\mathscr{B} = \varphi(\mathscr{A} \times I) \cup \psi(\mathscr{A} \times I) \subset \mathrm{Pol}(X)$ with $\operatorname{diam} \mathscr{B} < \varepsilon$. Observe that each $B \in \mathscr{B}$ contains some $A \in \mathscr{A}$ and each component of B meets A. □

Hereafter we assume that X *has no free arc.*

3.2. **Lemma.** $C^{\mathrm{Pol}}(X)$ *is a* Z_σ*-set in* $C(X)$.

Proof. By Lemma 2.1, $C^{\mathrm{Pol}}(X)$ is σ-compact, so F_σ in $C(X)$. Then it suffices to show that for each $\varepsilon > 0$, there is a map $\varphi : C(X) \to C(X) \setminus \mathrm{Pol}(X)$ with $\rho_H(\varphi, \mathrm{id}) < \varepsilon$. Subdividing K, we may assume that $\operatorname{mesh} K < \varepsilon/2$. Let $\{\tau_i | i = 1, \dots, n\}$ be the set of principal (= maximal dimensional) simplexes of K. Then $X = \bigcup_{i=1}^n \tau_i$. Choose Euclidean balls B_i in $\dot{\tau}_i = \operatorname{int} \tau_i$ such that $\dim B_i = \dim \tau_i \geq 2$ and set $Y = X \setminus \bigcup_{i=1}^n \dot{B}_i$. Since $C(\partial B_i)$ is an AR, we

have a map $f_i : B_i \to C(\partial B_i)$ such that $f_i|\partial B_i = \mathrm{id}$. Let $f : X \to C(Y)$ be the map defined by $f|Y = \mathrm{id}$ and $f|B_i = f_i$. Then f induces the map $\theta : C(X) \to C(Y) \subset C(X)$ defined by $\theta(A) = \bigcup\{f(a)|a \in A\}$ (cf. [Ke]). It is easy to see that $\rho_H(\theta, \mathrm{id}) < \varepsilon/2$. If A contains some B_i, then $\theta(A)$ contains ∂B_i but misses $\dot{B}_i$ whence any point of ∂B_i has no cone neighborhood in $\theta(A)$ (because $\dim B_i \geq 2$), that is, $\theta(A)$ is not a polyhedron (cf. [RS]). Let $\kappa : C(X) \to C(X)$ be the map defined by $\kappa(A) = \{x \in X|\rho(x, A) \leq \varepsilon/2\}$. Clearly $\rho_H(\kappa, \mathrm{id}) \leq \varepsilon/2$ and each $\kappa(A)$ contains some B_i. Thus we have the map $\varphi = \theta \circ \kappa : C(X) \to C(X) \setminus \mathrm{Pol}(X)$ with $\rho_H(\varphi, \mathrm{id}) < \varepsilon$. □

For each $n \in \mathbf{N}$, let $Y_n = |(\mathrm{Sd}^{2n} K)^1|$ and

$$P_n = \{A \in C(Y_n)|\exists \tau \in \mathrm{Sd}^{2n} K \text{ s.t. } \dim \tau = 1 \text{ and } \tau \subset A\}.$$

Then each P_n is closed in $C(Y_n)$, and is therefore fd compact.

3.3. **Lemma.** *For each $n \in \mathbf{N}$, there exists an embedding $\xi : P_n \times I \to P_{n+1}$.*

Proof. We use the path length metric ρ' on Y_{n+1}. One should note that ρ' is not the restriction of ρ. Let

$$\begin{aligned}\beta &= \min\{\rho'(v, v')|v \neq v' \in (\mathrm{Sd}^{2n+2} K)^0\}\\ &\leq \frac{1}{4} \cdot \min\{\rho'(v, v')|v \neq v' \in (\mathrm{Sd}^{2n} K)^0\}.\end{aligned}$$

Then for points x and x' in any connected subgraph Z of Y_{n+1}, if $\rho'(x, x') < \beta$ then $\rho'(x, x')$ is the distance between x and x' with respect to the path length metric on Z. Let $\kappa : P_n \times I \to P_n$ be the map defined by $\kappa(A, t) = \{x \in Y_n|\rho'(x, A) \leq t\beta\}$. For each $(A, t) \in P_n \times I$ and $\tau \in \mathrm{Sd}^n K$ with $\dim \tau = 1$, if $t\beta \geq \frac{1}{2} \cdot \mathrm{diam}(\tau \setminus A)$ then $\tau \cap A \neq \varnothing$ and $\hat{\tau} \in \kappa(A, t)$, where $\mathrm{diam}\,\varnothing = 0$. Then we can define a map $\eta : P_n \times I \to P_{n+1}$ as follows:

$$\begin{aligned}\eta(A, t) = \kappa(A, t) \cup \{x \in |\mathrm{St}(\hat{\tau}, \mathrm{Sd}^{2n+1} K)^1| \,|\, \tau \in \mathrm{Sd}^{2n} K,\\ \dim \tau = 1,\ \rho'(x, \hat{\tau}) \leq t\beta - \frac{1}{2} \cdot \mathrm{diam}(\tau \setminus A)\}.\end{aligned}$$

And we define $\zeta : P_n \times I \to P_{n+1}$ as follows:

$$\begin{aligned}\zeta(A, t) = A \cup \{x \in |\mathrm{St}(v\hat{\tau}, \mathrm{Sd}^{2n+2} K)^1| \,|\, v \lneqq \tau \in (\mathrm{Sd}^{2n} K)^1,\\ \tau \setminus A \neq \varnothing,\ 0 < \rho'(x, v) \leq t \cdot \delta(A, \tau, v)\},\end{aligned}$$

where

$$\delta(A, \tau, v) = \frac{\rho'(v, \tau \setminus A) \cdot \mathrm{diam}(\tau \setminus A) \cdot \beta}{(\mathrm{diam}\,\tau)^2} < \beta.$$

It is straightforward to see that ζ is continuous. The desired embedding $\xi : P_n \times I \to P_{n+1}$ can be defined by $\xi(A, t) = \eta(A, t) \cup \zeta(A, t)$. In fact, each $A \in P_n$ contains some 1-simplex $\tau \in \mathrm{Sd}^n K$ and then

$$\rho'(\hat{\tau}, |\mathrm{St}(\hat{\tau}, \mathrm{Sd}^{2n+1} K)^1| \setminus \xi(A, t)) = t\beta.$$

Hence $t \neq t'$ implies $\xi(A, t) \neq \xi(A', t')$ for any $A, A' \in P_n$. If $A \neq A' \in P_n$ then $\dot{\tau} \cap A \neq \dot{\tau} \cap A'$ for some 1-simplex $\tau \in \mathrm{Sd}^n K$. In case $\mathrm{diam}(\tau \setminus A) \neq \mathrm{diam}(\tau \setminus A')$, $\eta(A, t) \neq \eta(A', t)$. In case $\mathrm{diam}(\tau \setminus A) = \mathrm{diam}(\tau \setminus A')$ (> 0), $\delta(A, \tau, v) \neq \delta(A', \tau, v)$ for at least one vertex v of τ, so $\zeta(A, t) \neq \zeta(A', t)$. Thus $A \neq A'$ implies $\xi(A, t) \neq \xi(A', t)$ for each $t \in [0, 1]$. Therefore ξ is an embedding. By definition, $\xi(A, 0) = A \subset Y_n$ for each $A \in P_n$. □

3.4. **Lemma.** *$\bigcup_{n \in \mathbf{N}} P_n$ is dense in $C(X)$.*

Proof. For each $A \in C(X)$ and $\varepsilon > 0$, choose $n \in \mathbf{N}$ so that $\mathrm{mesh}(\mathrm{Sd}^{2n} K) < \varepsilon$ and set $B = Y_n \cap \bigcup\{\tau \in \mathrm{Sd}^{2n} K | A \cap \tau \neq \emptyset\} \in P_n$. Then $\rho_H(A, B) < \varepsilon$. Therefore $\bigcup_{n \in \mathbf{N}} P_n$ is dense in $C(X)$. □

3.5. **Lemma.** *$\bigcup_{n \in \mathbf{N}} P_n$ is an* fd-cap *set for $C(X)$.*

Proof. By Lemma 3.2, each P_n is an fd compact Z-set in $C(X)$. Let $A \supset B$ be fd compacta, $f : A \to C(X)$ a map such that $f|B : B \to P_n$ is an embedding into some P_n, and let $\varepsilon > 0$. Note that in Lemma 3.1 $\mathscr{B} \subset P_m$ if $\mathscr{A} \subset P_m$. Similar to [GH, Lemma 2], by using Lemmas 3.4 and 3.1, we can construct a map $g : A \to P_m$ of A into some $P_m \supset P_n$ such that $\rho_H(g, f) < \varepsilon/2$ and $g|B = f|B$. As in the proof of [CN, Lemma 4.6], by using Lemma 3.3 we can replace g by an embedding $h : A \to P_l$ of A into some $P_l \supset P_m$ such that $h|B = g|B = f|B$ and $\rho_H(h, g) < \varepsilon/2$ so $\rho_H(h, f) < \varepsilon$. Therefore $\bigcup_{n \in \mathbf{N}} P_n$ is an fd-cap set for $C(X)$. □

Now we complete the proof of the second part of the Main Theorem. Since $\bigcup_{n \in \mathbf{N}} P_n \subset C^{\mathrm{Pol}}(X)$, the result follows from Lemmas 3.5, 2.1, and 3.2, and [Ch, Lemma 4.2]. □

4. Star hyperspaces

Recall that K is a finite connected simplicial complex with $|K| = X$. In [CS_2], Curtis and Schori introduced the *star hyperspace* 2^K_{sst} of K by restricting the size of compacta in 2^X as follows. For each $v \in K^0$, let

$$\mathrm{sst}(v) = \bigcup\{\tau \in \mathrm{Sd}^2 K | \, |\mathrm{St}(v, \mathrm{Sd}\, K)| \cap \tau \neq \emptyset\}.$$

We define the following star hyperspaces.

$$\begin{aligned}
&2^K_{\mathrm{sst}} = \{A \in 2^X | A \subset \mathrm{sst}(v) \text{ for some } v \in K^0\}; \\
&C_{\mathrm{sst}}(K) = C(X) \cap 2^K_{\mathrm{sst}}; \qquad C_{\omega\,\mathrm{sst}}(K) = C_\omega(X) \cap 2^K_{\mathrm{sst}}; \\
&\mathrm{Pol}_{\mathrm{sst}}(K) = \mathrm{Pol}(X) \cap 2^K_{\mathrm{sst}}; \qquad C^{\mathrm{Pol}}_{\mathrm{sst}}(K) = C^{\mathrm{Pol}}(X) \cap 2^K_{\mathrm{sst}}; \\
&\mathscr{F}_{\mathrm{sst}}(K) = \mathscr{F}(X) \cap 2^K_{\mathrm{sst}}.
\end{aligned}$$

In [CS_2], Curtis and Schori proved that $2^K_{\mathrm{sst}} \cong X \times Q$ and that $C_{\mathrm{sst}}(K) \cong X \times Q$ if X has no free arcs. This is generalized as follows.

4.1. **Theorem.** *For any finite connected simplicial complex K,*

$$(2^K_{\mathrm{sst}}, \mathscr{F}_{\mathrm{sst}}(K)) \cong (|K| \times Q, |K| \times \sigma),$$
$$(2^K_{\mathrm{sst}}, \mathrm{Pol}_{\mathrm{sst}}(K)) \cong (|K| \times Q, |K| \times \sigma).$$

In case K has no principal 1-simplex,

$$(2^K_{\mathrm{sst}}, C_{\omega\,\mathrm{sst}}(K)) \cong (|K| \times Q, |K| \times \Sigma),$$
$$(C_{\mathrm{sst}}(K), C^{\mathrm{Pol}}_{\mathrm{sst}}(K)) \cong (|K| \times Q, |K| \times \sigma).$$

This theorem is valid for a locally finite simplicial complex K without isolated vertices. To prove the theorem, we need the relative version of [CS_2, Theorem 3.5]. A locally finite closed cover $\mathscr{D} = \{Q_i\}$ of a metric space M is called a *Q-decomposition* if (i) each Q_i is homeomorphic to Q; (ii) $\mathscr{D}$ contains all nonempty intersections of its members; and (iii) if Q_j is a proper subset of Q_k, then Q_j is a Z-set in Q_k. We say that Q-decompositions $\{Q_i\}$ and $\{Q'_i\}$ of spaces M and M' are *isomorphic* if they are order isomorphic with respect to the partial orderings given by set inclusion. It is straightforward to see the following version of [CS_2, Theorem 3.5].

4.2. **Theorem.** *Let $\{Q_i\}$ and $\{Q'_i\}$ be isomorphic Q-decompositions of spaces M and M', $N \subset M$ and $N' \subset M'$ such that each $Q_i \cap N$ and $Q'_i \cap N'$ are* fd-cap *sets (or cap sets) for Q_i and Q'_i, respectively. Then there exists a homeomorphism of M onto M' taking each element of $\{Q_i\}$ onto the corresponding element of $\{Q'_i\}$ and N onto N'.*

Proof of Theorem 4.1. For $A_1, \dots, A_m \in 2^X$, let

$$2^X(A_1, \dots, A_m) = \{A \in 2^X | A \cap A_i \neq \varnothing \text{ for each } i\},$$
$$C(X; A_1, \dots, A_m) = C(X) \cap 2^X(A_1, \dots, A_m).$$

First, by [CN, Corollary 5.2], $\mathscr{F}(X; A_1, \dots, A_m) = \mathscr{F}(X) \cap 2^X(A_1, \dots, A_m)$ is an fd-cap set for $2^X(A_1, \dots, A_m)$. As in §2, we can see that

$$C_\omega(X; A_1, \dots, A_m) = C_\omega(X) \cap 2^X(A_1, \dots, A_m)$$

is a Z_σ-set in $2^X(A_1, \dots, A_m)$, so

$$\mathrm{Pol}(X; A_1, \dots, A_m) = \mathrm{Pol}(X) \cap 2^X(A_1, \dots, A_m)$$

is also an fd-cap set for $2^X(A_1, \dots, A_m)$. Similar to [CS_2, Theorem 4.3], the first half of the theorem follows from Theorem 4.2.

In the case where X has no free arcs, $C_\omega(X; A_1, \dots, A_m)$ is a cap set for $2^X(A_1, \dots, A_m)$ by the result of [Mi_2]. By choosing B_i in the proof of Lemma 3.2 so that $B_i \cap A_j = \varnothing$ or $B_i \subsetneqq A_j$ for each $j = 1, \dots, m$, we can prove that

$$C^{\mathrm{Pol}}(X; A_1, \dots, A_m) = C^{\mathrm{Pol}}(X) \cap 2^X(A_1, \dots, A_m)$$

is a Z_σ-set for $C(X; A_1, \dots, A_m)$. Now we assume that each A_i is a polyhedron triangulated by a subcomplex of $\mathrm{Sd}^2 K$. Then each A_i meets each Y_n defined in §3. For each $n \in \mathbf{N}$, let $P_n(A_1, \dots, A_m) = P_n \cap 2^X(A_1, \dots, A_m)$, where P_n is defined in §3. Then we can prove, in a way similar to Lemma 3.5, that $\bigcup_{n\in\mathbf{N}} P_n(A_1, \dots, A_m)$ is an fd-cap set for $C(X; A_1, \dots, A_m)$. Since $\bigcup_{n\in\mathbf{N}} P_n(A_1, \dots, A_m) \subset C^{\mathrm{Pol}}(X; A_1, \dots, A_m)$, $C^{\mathrm{Pol}}(X; A_1, \dots, A_m)$ is an fd-cap set for $C(X; A_1, \dots, A_m)$. Therefore the second half also follows from Theorem 4.2. □

5. Remarks

In the case where X is a finite graph, we have $\mathrm{Pol}(X) = C_\omega(X)$ and $C^{\mathrm{Pol}}(X) = C(X)$. The first assertion of the theorem has been shown in this case by Michael [Mi_1] (cf. [Mi_2]), and the assumption of the second statement is essential.

The second part of the Main Theorem can be generalized as follows.

5.1. **Theorem.** *If X is a compact connected polyhedron with no free arc, then for each $n \in \mathbf{N}$, $(C_n(X), C_n(X) \cap \mathrm{Pol}(X)) \cong (Q, \sigma)$.*

Proof. First note that in Lemma 3.1, $\mathscr{B} \subset C_n(X) \cap \mathrm{Pol}(X)$ if $\mathscr{A} \subset C_n(X) \cap \mathrm{Pol}(X)$. In the proof of Lemma 3.2, the map φ can be defined on 2^X. In this case, we have $\varphi(C_n(X)) \subset C_n(X)$ for each $n \in \mathbf{N}$. Hence it follows that $C_n(X) \cap \mathrm{Pol}(X)$ is also a Z_σ-set in $C_n(X)$ for each $n \in \mathbf{N}$ if X has no free arc. As is easily seen, $C_n(X) \cap \mathrm{Pol}(X)$ is dense in $C_n(X)$. For each $m, n \in \mathbf{N}$, let $Y_m = |(\mathrm{Sd}^m K)^1|$ and $P_{n,m} = \{A \in C_n(Y_m) | A$ contains some $\tau \in (\mathrm{Sd}^m K)^1\}$. As in §3, we can prove that $\bigcup_{m\in\mathbf{N}} P_{n,m}$ is an fd-cap set for $C_n(X)$. For each $n \in \mathbf{N}$, $C_n(X) \cong Q$ and $\bigcup_{m\in\mathbf{N}} P_{n,m} \subset C_n(X) \cap \mathrm{Pol}(X)$. (E.g., $C_n(X) \cong Q$ follows from [Cu, Corollary 5.1] since it is as easy to show as in Lemma 3.2 that $\{X\}$ is a Z-set in $C_n(X)$.) Thus we have the result similar to the second part of the Main Theorem. □

From Theorem 5.1, it is natural to conjecture as follows.

5.2. **Conjecture.** For a compact connected polyhedron X with no free arc, $(2^X, C_\omega(X), \mathrm{Pol}(X)) \cong (Q, \Sigma, \sigma)$.

Applying the characterization of the triple (Q, Σ, σ) in [SW], this conjecture is reduced to show that each $C_n(X)$ is a Z-set in some $C_m(X)$ (cf. [Sa]). And related to the result of §4, it is also natural to conjecture as follows.

5.3. **Conjecture.** For a (locally) finite simplicial complex K with no principal 1-simplexes,

$$(2^K_{\mathrm{sst}}, C_{\omega\,\mathrm{sst}}(K), \mathrm{Pol}_{\mathrm{sst}}(K)) \cong (|K| \times Q, |K| \times \Sigma, |K| \times \sigma).$$

One may also conjecture that $(2^X, C_\omega(X), \mathscr{F}(X)) \cong (Q, \Sigma, \sigma)$ since $\mathscr{F}(X) \subset C_\omega(X) \subset 2^X$, $(2^X, C_\omega(X)) \cong (Q, \Sigma)$, and $(2^X, \mathscr{F}(X)) \cong (Q, \sigma)$. However

this conjecture is false. In fact, $C_\omega(X)\setminus\mathscr{F}(X) = \bigcup_{n\in\mathbf{N}} C_n(X)\setminus\mathscr{F}(X)$ and each $C_n(X)\setminus\mathscr{F}(X)$ is σ-compact because $\mathscr{F}(X)\cap C_n(X) = \mathscr{F}_n(X)$ is compact, so it is closed in $C_n(X)$. Hence $C_\omega(X)\setminus\mathscr{F}(X)$ is also σ-compact. On the other hand, $\Sigma\setminus\sigma$ is not σ-compact because $\Sigma\setminus\sigma \cong \Sigma\times s$ by [SW, Theorem 3.1]. This observation is due to Jan van Mill. The author would like to thank Jan van Mill for his observation.

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