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FINITE COVERINGS BY COSETS OF NORMAL SUBGROUPS

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ABSTRACT. In this brief note, we characterize those groups G which can be covered by finitely many cosets a_iM_i of maximal normal subgroups M_i, where the covering is irredundant and not all M_i are equal. This refines an earlier result of Brodie, Chamberlain, and Kappe, who characterized those groups which can be covered by finitely many proper normal subgroups.

A group G is said to be covered by a collection of cosets of subgroups if each element of the group belongs to at least one of the cosets. The covering is said to be irredundant if each of the cosets contains at least one element which belongs to no other coset.

The case where each coset in a covering is actually a normal subgroup has been investigated in a recent paper by Brodie, Chamberlain, and Kappe [3]. One of their main results is the following theorem.

Theorem 1. *A group can be covered by finitely many proper normal subgroups if and only if it has a quotient isomorphic to an elementary abelian p-group of rank two for some prime p.*

B. H. Neumann [6] had previously shown that a group can be covered by finitely many proper subgroups (not necessarily normal) if and only if the group has a finite noncyclic quotient.

Various questions involving coverings by cosets which are not necessarily subgroups have been studied by several authors—Berger, Felzenbaum, and Fraenkel [1], or Parmenter [7], for example. Much of the impetus for these papers comes from number theory, where a great deal of work has been done on problems concerning covering the integers by sets of arithmetic progressions (see [2] or [8]).

In this paper, we seek to extend Theorem 1, as stated above, to coverings by cosets of normal subgroups. As a by-product of this investigation, we obtain a different (perhaps simpler) proof of Theorem 1 than that given in [3].

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Since a group can always be covered by the complete set of cosets of any subgroup, some condition is needed to avoid this situation. Unfortunately, these trivial coverings are not always immediately recognizable, as can be seen in the following example given by Herzog and Schönheim [4]. Let $G = \langle x_1, x_2, x_3, x_4\rangle$ be the elementary abelian group of order 16. Observe that G is the union of the 5 cosets $\langle x_2, x_3, x_4\rangle$, $\langle x_2\rangle + x_1 + x_4$, $\langle x_3\rangle + x_1 + x_2$, $\langle x_4\rangle + x_1 + x_3$, $\langle x_2 + x_3 + x_4\rangle + x_1$, and that all the subgroups are different. However, the union of the last four cosets is just $\langle x_2, x_3, x_4\rangle + x_1$, so we really have a trivial covering here, but in disguised form.

Note, however, that a group G has a finite covering by subgroups if and only if it has a finite covering by maximal subgroups, and if our attention is restricted to the latter setting, the following result is obtained.

Theorem 2. *For a group G, the following are equivalent:*

(i) *G is covered by a finite set of cosets a_iM_i, with M_i a maximal normal subgroup of G for all i, such that the covering is irredundant and not all M_i are equal.*

(ii) *G has a quotient isomorphic to an elementary abelian p-group of rank two for some prime p.*

As in all work of this type, the following result, due to B. H. Neumann, is crucial.

Lemma [5]. *Let G be covered by a finite set of cosets a_iG_i. If we omit from this covering any coset a_iG_i for which $[G\colon G_i]$ is infinite, then G is still covered by the remaining cosets.*

We now proceed to prove our result.

Proof of Theorem 2. To prove (ii) $\Rightarrow$ (i), note that (ii) easily implies that G can be covered by finitely many proper normal subgroups, and as remarked earlier this leads to (i).

Assume (i). By the lemma, and the fact that the covering is irredundant, we know that $[G\colon M_i] < \infty$ for all i. Passing to the finite quotient G/M, where $M = \bigcap M_i$, we may assume $M = 1$.

Now select any j and let $S = \{i | M_i \neq M_j\}$. Because of our assumption that not all M_i are equal, S is nonempty.

We know already that $\bigcap M_i = 1$, and we will now show that $\bigcap_{i\in S} M_i = 1$. Suppose this is not true and let $B = \bigcap_{i\in S} M_i \neq 1$. Because M_j is a maximal normal subgroup of G and $B \cap M_j = 1$, we know that $G = B \times M_j$. It follows that, for each coset of M_j present in the covering of G, we can assume that the coset representative is in B. We may also assume, by multiplying all cosets in the covering by a group element if necessary, that M_j, the coset of 1, is part of our covering. Finally, note that some dM_j, where $d \neq 1$ is in B, is not part of the covering.

Since the covering is irredundant, there exists m in M_j which does not belong to any other coset in the covering. But it is then easy to see that dm cannot belong to any of the cosets in the covering (note that, by the above, d is contained in every $M_i \neq M_j$, and $dm = md$ since $B \cap M_j = 1$), and this gives our contradiction. We conclude that $B = 1$.

Next, choose L, a set of minimal cardinality such that $\bigcap_{i \in L} M_i = 1$. Note that if $i \neq j$, with i and j in L, then M_i and M_j are distinct. From the previous work, we know that there exists M_ℓ such that $M_\ell \neq M_i$ for all i in L.

Let j be any member of L, and let A be the intersection of the sets M_i, where i is in L but $i \neq j$. By minimality, we know that $A \neq 1$ and $A \bigcap M_j = 1$. Since G/M_j is simple, $G = AM_j$, and this means that $A \simeq G/M_j$ is simple. Now $M_\ell \neq M_j$ allows us to choose $z = ab$ in M_ℓ, with a in A, $a \neq 1$, and b in M_j.

We claim that the simple group A must be abelian. Assume not; then there exists c in A such that $cac^{-1} \neq a$. Now $M_\ell \triangleleft G$ implies czc^{-1} is in M_ℓ, and $czc^{-1} = cac^{-1}cbc^{-1} = cac^{-1}b$ since $A \bigcap M_j = 1$, so $czc^{-1}z^{-1} = cac^{-1}a^{-1} \neq 1$. It follows that $M_\ell \bigcap A \neq 1$. Since A is simple, this means that $A \subseteq M_\ell$. Because this is true for all M_ℓ of the type described, we conclude that A must be equal to the subgroup B defined earlier. Since we saw that this latter subgroup is trivial, there is a contradiction. Thus A must be abelian, i.e., G/M_j is cyclic of prime order.

Since the above argument works for all j in L and $\bigcap_{j \in L} M_j = 1$, we conclude that G is abelian. If the primes $[G\colon M_j]$ were all distinct, G would by cyclic of order $n = p_1 p_2 \cdots p_t$ (where the p_i are the distinct primes). But in this case, a coset of a maximal subgroup is just a congruence modulo p_i for some i. A covering of the type described in (i) must omit at least one congruence $x \equiv a_i \pmod{p_i}$ for each i, and the Chinese Remainder Theorem then tells us that some element of G is not covered.

Hence, two of the primes $[G\colon M_j]$ must be equal, and we have the result. □

Remarks. 1. If G is covered by finitely many proper normal subgroups, then G must satisfy condition (i), as remarked earlier. Hence Theorem 1 can be derived as a corollary of Theorem 2, giving an alternative approach to the proof of Theorem 1 from that given in [3]. In any case, Theorems 1 and 2 together tell us that any group which satisfies condition (i) can also be covered by finitely many proper normal subgroups.

2. Theorem 2 is clearly not true if the word "maximal" is omitted. To see this in a trivial fashion, let G be any finite group with proper normal subgroups M, N where N is strictly contained in M—first express G as a union of all cosets of M, and then decompose one of these cosets into a union of cosets of N.

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